adamjee maths

A-LEVEL MATHS / EDEXCEL / FREE RESOURCES

Differential equations

Mark scheme

8 questions

Exam questions and marking instructions remain credited to their board.

Contents

1 June 2025 - Paper 2 - Q10

2 June 2024 - Paper 1 - Q7

3 June 2024 - Paper 1 - Q14

4 June 2024 - Paper 2 - Q12

5 June 2022 - Paper 2 - Q14

6 October 2020 - Paper 1 - Q14

7 June 2019 - Paper 2 - Q14

8 June 2018 - Paper 1 - Q10

Question	Scheme	Marks	AOs
10(a)	$\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$	M1	3.1b
	$\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ $20\frac{dV}{dt} = 9 - 6V$ *	A1*	2.1
		(2)	
(b)	e.g. $\frac{1}{9-6V}\frac{dV}{dt} = \frac{1}{20} \rightarrow \int \frac{1}{9-6V}dV = \int \frac{1}{20}dt$	B1	1.1b
	$\frac{1}{9-6V} \rightarrow \ldots \ln\lvert 9-6V\rvert$	M1	1.1b
	$-\frac{1}{6}\ln\lvert 9-6V\rvert = \frac{t}{20} \quad (+c)$	A1	1.1b
	$-\frac{1}{6}\ln\lvert 9-6V\rvert = \frac{t}{20} + c$ $9-6V = Ae^{-\frac{3t}{10}}$ $t = 0, V = 0.25 \Rightarrow A = (7.5)$ **or** $-\frac{1}{6}\ln\lvert 9-6V\rvert = \frac{t}{20} + c$ $t = 0, V = 0.25 \Rightarrow c = \left(-\frac{1}{6}\ln 7.5\right)$	dM1	3.1a
	$9-6V = 7.5e^{-\frac{3t}{10}}$ $V = \frac{3}{2} - \frac{5}{4}e^{-\frac{3t}{10}}$ **or** $\frac{1}{6}\ln 7.5 - \frac{1}{6}\ln\lvert 9-6V\rvert = \frac{t}{20}$ $\ln\frac{7.5}{9-6V} = 0.3t$ $9-6V = 7.5e^{-0.3t}$ $V = \frac{3}{2} - \frac{5}{4}e^{-0.3t}$	A1	2.1
		(5)	

Notes

(a) **Marks for part (a) may not be scored in part (b)**

M1: Either $\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$ o.e. e.g. $\frac{dV}{dt} = \frac{9}{20}$ or $\frac{dV}{dt} = \pm\frac{3}{10}V$ seen or implied by e.g. $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ (but not implied by just stating the given answer). Condone use of $\dot{V}$

It may be seen as part of their $\frac{dV}{dt}$ e.g. $\frac{dV}{dt} = 0.45 + V + 0.3V$ scores M1A0*

Condone e.g. change in volume = (inflow – outflow =) $0.45 - 0.3V$ for this mark.

A1*: Achieves $20\frac{dV}{dt} = 9 - 6V$ with no errors, following $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ o.e. (including the $\frac{dV}{dt}$ or $\dot{V}$ but note that it must be $\frac{dV}{dt}$ in the final line and not $\dot{V}$).

change in volume = $0.45 - 0.3V \rightarrow 20\frac{dV}{dt} = 9 - 6V$ scores M1A0*.

Ignore any units used in their working for both marks.

(b)

B1: Separates the variables correctly, e.g., $\int \frac{1}{9-6V}dV = \int \frac{1}{20}dt$ or $\int \frac{20}{9-6V}dV = \int \{1\}\,dt$ o.e.

The integral symbol and/or dV and/or dt may be implied if they go on to integrate **both** sides to the correct form $\ldots\ln\lvert\alpha(9-6V)\rvert = \ldots t \quad (+c)$ with or without the modulus brackets.

M1: Attempts to integrate the reciprocal term $\frac{\beta}{9-6V} \to \ldots \ln\left|9-6V\right|$ or $\to \ldots \ln\left|\alpha\left(6V-9\right)\right|$ for some constant β (and α if used). Condone e.g. $\frac{20}{9-6V} \to \ldots \ln 9-6V$ or $\to \ldots \ln 6V-9$

A1: Correct integration for both sides. They do not need the + c for this mark.
Note scoring this mark implies the earlier B1 (unless it is a verification attempt – see SC).
Note that e.g. $-\frac{1}{6}\ln\left|3-2V\right| = \frac{t}{20} \ \ (+c)$ or $-\frac{10}{3}\ln\left|2V-3\right| = t \ \ (+c)$ are also correct.
$-\frac{1}{6}\ln\left(9-6V\right) = \frac{t}{20} \ (+c)$ is also correct.
Condone log being used in place of ln.

dM1: Requires constant of integration now. Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for c, which may be "A" = e^c if they rearrange first to eliminate ln terms.
Dependent on the previous method mark.
Do not be concerned about their processing to find c or "A" = e^c and does not need to be exact.

A1: Achieves the required form e.g. $V = \frac{3}{2} - \frac{5}{4}e^{-\frac{3t}{10}}$ with no errors and clear working.
Allow equivalent fractions or decimals e.g. $V = 1.5 - 1.25e^{-\frac{6}{20}t}$

SC: Attempts by verification may score maximum B0M1A1dM1A0 – see below.
Alt: Use of an integrating factor – see below.

(b) **Special Case: Attempts by verification may score maximum B0M1A1dM1A0**

B0: This mark may not be scored via this approach.

M1: Differentiates $V = P - Qe^{-kt}$ to the form $\frac{dV}{dt} = \alpha e^{-kt}$ where α is a constant (note it should be $\frac{dV}{dt} = Qke^{-kt}$) **and** substitutes both this and $V = P - Qe^{-kt}$ into $20\frac{dV}{dt} = 9 - 6V$ **and** deduces a value for P **or** k by comparing coefficients.

A1: Correct values for both P **and** k.

dM1: Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for Q.
Requires a value for P to have been found using the above approach.

A0: This mark may not be scored via this approach.

Alternative: Using Integrating Factor (Further Maths)

B1: Deduces the correct integrating factor for the equation, $e^{0.3t}$
This should come from $\frac{dV}{dt} + 0.3V = 0.45 \Rightarrow \text{I.F.} = e^{\int 0.3dt} = e^{0.3t}$
May be implied by sight of $\frac{d\left(Ve^{0.3t}\right)}{dt} = \ldots$

M1: Fully multiplies through by their integrating factor and integrates both sides.
Score for $Ve^{kt} = \int \ldots e^{kt} dt = \ldots e^{kt}$ Condone missing dt

A1: Correct integration $Ve^{0.3t} = \int 0.45e^{0.3t} dt = \frac{3}{2}e^{0.3t} \ (+c)$

dM1: As main scheme.

A1: As main scheme.

Question	Scheme	Marks	AOs
10(c)	**Examples:** (1) $\frac{dV}{dt}=0\Rightarrow V=(1.5)$ (or e.g. max V is 1.5) (2) As $t\to\infty$, $e^{-"0.3"t}\to 0$ (or $V\to$ "1.5") (3) Flow in = flow out at max V so $0.3V=0.45\Rightarrow V=1.5$ (4) As $e^{-"0.3"t}>0$, $V<"1.5"$ (5) When $V>1.5$, $\frac{dV}{dt}<0$ (6) $V=2\Rightarrow\frac{dV}{dt}=-0.15$ **or** compares $\frac{dV_{out}}{dt}\ (=0.6)$ against $\frac{dV_{in}}{dt}\ (=0.45)$ at $V=2$ (7) $V=2\Rightarrow "1.5"-"1.25"e^{-"0.3"t}=2\Rightarrow e^{-"0.3"t}<0$ (8) $"1.5"-"1.25"e^{-"0.3"t}=2\Rightarrow \ln(-0.4)$ is undefined (condone e.g. gives a maths error)	M1	3.2a
	• The (upper) limit for V is 1.5 (m^3) so no (the container will not become full) (first 4 bullets) • If $V=2$ (or $V>1.5$), it would be emptying so no (it can never be full) (bullets 5, 6) • No, as the equation cannot be solved (or is not true/doesn't work) when $V=2$ (bullets 7, 8)	A1ft	2.4
		(2)	
			(9 marks)

Notes

(c)

M1: See main scheme. If using the answer to part (b) it must be of the form $V=P-Qe^{-kt}$ but there is no limitation on the values of their P, Q or k.
Substitution of a large value for t may score this mark but it is unlikely to be recovered to score the A1 unless they reference e.g. V_{max} being "1.5".
Reference to an (upper) limit of "1.5" or their P can imply the method mark.
If setting $V=2$ in their equation they must reach either ln(–ve) **or** solve the equation to reach a value for t to score this mark.

A1ft: Must conclude "no" or equivalent e.g. "the container will not become full".
Makes a correct interpretation for their method (see bullets 1-8) with a clear conclusion e.g. "no".
To score this mark through ft, their V must be of the form $V=P-Qe^{-kt}$ with $k>0$, $Q>0$ **and** $0<P<2$ if used (but note that they can still use the answer to part (a) to score both marks via bullets 1, 3, 5 or 6). Allow "it" in place of the "container"/"tank".
Just stating "the equation cannot be solved when $V=2$" without any evidence is M0A0.
There must be no incorrect working if solving their equation or contradictory statements such as "t cannot be negative" but condone notational errors provided the intention is clear.

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
7(a)	$\{H =\} 0.6\mathrm{e}^{-0.2t} \{+c\}$	M1	1.1b
	$t = 0, H = 1.5 \Rightarrow 1.5 = "0.6" + c$ $\Rightarrow c = 0.9$	dM1	3.4
	$\Rightarrow H = 0.6\mathrm{e}^{-0.2t} + 0.9$	A1	2.1
		(3)	
(b)	$1.2 = 0.6\mathrm{e}^{-0.2t} + 0.9 \Rightarrow 0.6\mathrm{e}^{-0.2t} = 0.3$	M1	3.4
	$\mathrm{e}^{-0.2t} = \frac{1}{2}$ $\Rightarrow t = -5\ln\left(\frac{1}{2}\right)$	dM1	1.1b
	$\{t =\}$ 3 hours 28 minutes	A1	3.2a
		(3)	
(c)	{As t gets large $H \rightarrow$} 0.9	M1	3.1b
	0.9 m or 90 cm	A1ft	2.2b
		(2)	
			(8 marks)

Notes

(a)

M1: Attempts integration to achieve $\{H =\} k\,\mathrm{e}^{-0.2t} \{+c\}$ with k a numerical constant $\neq -0.12$
Note that we will condone $k = (-0.2)(-0.12)$ $\{= 0.024\}$
If they divide by –0.12 first before integrating they need $aH = b\mathrm{e}^{-0.2t} \{+c\}$ with a and b numerical and $b \neq 1$. Condone a spurious integral symbol remaining after integration.

dM1: Uses $t = 0, H = 1.5$ and a model of the form $H = k\,\mathrm{e}^{-0.2t} + c$ (or $aH = b\mathrm{e}^{-0.2t} + c$) to find the value of the constant c. They cannot just "make up" a value for k.
Do not be concerned with their processing to find c but they cannot just state B (or c) is 0.
For reference if they divide by –0.12 first they should reach $-\frac{25}{3}H = -5\mathrm{e}^{-0.2t} - 7.5$ o.e.

A1: Correct complete equation in the required form: $H = 0.6\mathrm{e}^{-0.2t} + 0.9$ with the H = present.
May be awarded if seen at the start of (b) but not in (c). Condone $-\frac{1}{5}$ in place of –0.2
Finding correct values for A and B is insufficient for this mark.
Allow exact equivalents but they must be in the required form, e.g. $H = \frac{6}{10}\mathrm{e}^{-0.2t} + \frac{9}{10}$
A minimally acceptable answer is $\{H =\} 0.6\mathrm{e}^{-0.2t} + c \rightarrow H = 0.6\mathrm{e}^{-0.2t} + 0.9$ score M1dM1A1.
Note: sight of differentiating the given form to e.g. $\frac{\mathrm{d}H}{\mathrm{d}t} = -0.2A\mathrm{e}^{-0.2t}$ in their working without clear evidence of integration of the original differential equation should be marked using the special case below.

SC: For candidates starting with the given answer $H = A\mathrm{e}^{-0.2t} + B$ it is possible to use $\frac{\mathrm{d}H}{\mathrm{d}t} = -0.2A\mathrm{e}^{-0.2t} = -0.12\mathrm{e}^{-0.2t}$ to deduce that $A = 0.6$. This can be awarded SC M1dM0A0
If they go on to find B as in the main scheme then this can be awarded SC M1dM1A0
Answer with no working scores 110.

Note: If the special case is applied they may go on to achieve the rest of the marks in (b) and (c).

(b) Note: A and B must be numbers but may be "made up" if they did not have an answer to (a).

M1: Uses $H = 1.2$ in a model of the form $H = A\text{e}^{-0.2t} + B,\ B \neq 0$ and rearranges to make $A\text{e}^{\pm 0.2t}$ or $\text{e}^{\pm 0.2t}$ the subject. Condone slips in rearranging, e.g. dividing the LHS by 0.9 instead of subtracting 0.9. Rearranging first before substituting is acceptable but they must get to $A\text{e}^{\pm 0.2t}$ or $\text{e}^{\pm 0.2t}$ as the subject.

dM1: Correct use of ln to make t the subject. Requires $A > 0,\ 0 < B < 1.2$ **and** $\text{e}^{\pm 0.2t} = \lambda > 0$
If they had a negative value for A in part (a) they cannot just make it positive at this stage.
Any of $5\ln 2$ or $-5\ln\frac{1}{2}$ or awrt 3.46 or awrt 3.47 following a correct equation will imply M1dM1.
If they do not show their method for an incorrect $H = A\text{e}^{-0.2t} + B$ with $A > 0,\ 0 < B < 1.2$ you may need to check their value for $t > 0$ as it may imply M1dM1.

A1: Correct time in hours and minutes $\{t =\}\ 3\,\text{hours}\ 28\,\text{minutes},$ but condone e.g. 3h 28m
Must come from correct values of A and B in (a).

Note: If their $B = 0$ then they should end up with $t = -3.46\ldots$ however, they did not score the first M1. They cannot "recover" this by making it positive and finding $t = 3\,\text{hours}\ 28\,\text{minutes}$.

(c) Note that 0.9 or 0.9m must come from a correct value of B in (a) to score any marks.

M1: Identifies the requirement to establish the limit as t tends to infinity.
It can be implied by stating that $H = A\text{e}^{-0.2t} + B \to B$ or $\left(\lim_{t\to\infty}\left[0.6\text{e}^{-0.2t} + 0.9\right]\right) = 0.9$
Stating "B" on its own will score this mark.
Substituting a large value is M0 unless it leads to their value for B at which point the A1 is available as well.

A1ft: Correct height including units. Follow through on their value of B where $0 < B < 1.2$
Correct ft height including units implies M1A1, while e.g. 0.9 (no units) would imply M1A0.
Evidence of an incorrect method such as $1.5 - 0.6\text{e}^{-0.2(0)} = 0.9\,\text{m}$ scores M0A0.

Misreading as $\frac{\text{d}H}{\text{d}t} = -0.12\text{e}^{0.2t}$ can score a maximum (a) 110 (b) 100 (c) 10.

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
14(a)	$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{k}{\sqrt{r}}$	B1	3.3
		(1)	
(b)	$0.9=\frac{k}{\sqrt{16}}\Rightarrow k=3.6$	B1	3.4
	$\int \sqrt{r}\ \mathrm{d}r=\int "3.6"\mathrm{d}t\Rightarrow ...$	M1	2.1
	$\frac{2}{3}r^{\frac{3}{2}}="3.6"t\quad \{+c\}$	A1	1.1b
	$t=10, r=16\Rightarrow \frac{2}{3}\times 16^{\frac{3}{2}}=3.6\times 10+c\Rightarrow c=...$	dM1	3.4
	$r^{\frac{3}{2}}=5.4t+10$ *	A1*	1.1b
		(5)	
(c)	$t=20\Rightarrow r=\left(5.4(20)+10\right)^{\frac{2}{3}}=...$	M1	3.4
	$r=24.1$ cm	A1	1.1b
		(2)	
(d)	(The model will not hold indefinitely as) the balloon may burst	B1	3.5b
		(1)	
			(9 marks)

Notes

(a)

B1: Correctly sets up the model. $\frac{\mathrm{d}r}{\mathrm{d}t}\propto\frac{1}{\sqrt{r}}$ scores B0 unless e.g. $\frac{\mathrm{d}r}{\mathrm{d}t}=\pm\frac{k}{\sqrt{r}}$ is seen but condone $\frac{\mathrm{d}r}{\mathrm{d}t}=\pm\frac{k}{\sqrt{r}}$ being seen at the start of (b). They may use any letter except t or r in place of k.

You may see $\frac{\mathrm{d}r}{\mathrm{d}t}=\pm\frac{1}{k\sqrt{r}}$ which is acceptable provided it is clear that it is not the k^{th} root.

(b) **Note:** candidates using $\frac{\mathrm{d}r}{\mathrm{d}t}=\pm\frac{1}{\sqrt{r}}$ in (b) can score maximum B0M1A1dM0A0

B1: $k=3.6$ coming from $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{k}{\sqrt{r}}$ (or $k=\frac{5}{18}$ coming from $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{k\sqrt{r}}$) and from use of $r=16$ and $\frac{\mathrm{d}r}{\mathrm{d}t}=0.9$ but note that this may occur later in their working, which is perfectly fine provided it is from acceptable work. Note e.g. $k=-3.6$ coming from $\frac{\mathrm{d}r}{\mathrm{d}t}=-\frac{k}{\sqrt{r}}$ is correct.

Note, however, that an attempt to find k from comparing coefficients between $r^{\frac{3}{2}}=1.5kt+c$ and $r^{\frac{3}{2}}=5.4t+10$ is not acceptable (see special case).

They can also find k by differentiating **their** $r=\mathrm{f}(t)$ and substituting $t=10$, $r=16$ and $\frac{\mathrm{d}r}{\mathrm{d}t}=0.9$ **if** they have also used $t=10$, $r=16$ in **their** $r=\mathrm{f}(t)$. This sets up simultaneous equations where c can be eliminated. Use Review if you are unsure if their approach is acceptable.

M1: Separates the variables for their differential equation **correctly** and attempts to integrate both sides.

Must be a differential equation of the form $\frac{\mathrm{d}r}{\mathrm{d}t} = \mathrm{f}(r)$ for some function f(r) independent of t.

Evidence of $r^n \to r^{n+1}$, or e.g. $\frac{1}{r} \to \ln r$ is sufficient for their attempt to integrate in r, but k must be integrated to kt o.e. e.g. $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2} \Rightarrow r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = k \Rightarrow \lambda r^3 = kt\{+c\}$ would score this mark.

Note that they may divide by k (or 3.6) prior to integrating. Here, 1 must be integrated to t.

A1: Correct integration for their k. Allow this mark if they have not found k, so allow e.g. $\frac{2}{3}r^{\frac{3}{2}} = kt\{+c\}$ with/without the constant of integration but the $\frac{2}{3}$ must be evident in some way.

Condone spurious notation for this intermediate mark e.g. integral signs left in after integrating.

dM1: Uses $t = 10$, $r = 16$ in their equation to find the constant of integration.
This mark is dependent on the first method mark.
Must have already found a value for k using a valid strategy and the constant of integration must be present.

Those that found k from comparing coefficients between $r^{\frac{3}{2}} = \frac{3}{2}kt + c$ and $r^{\frac{3}{2}} = 5.4t + 10$ may not score this mark.

A1*: Correct equation from correct working. May be seen at the start of (c).
Must follow A1 earlier so do check if this has been obtained fortuitously.

SC: It is possible to compare coefficients (following integration) between $r^{\frac{3}{2}} = \frac{3}{2}kt + c$ and $r^{\frac{3}{2}} = 5.4t + 10$ to deduce the value of k as 3.6 (or just write their coefficient as 5.4).
The maximum that can be scored this way or by a similar invalid approach is B0M1A1dM0A0

(c)

M1: Substitutes $t = 20$ into the given equation and uses correct processing to find the value of r

e.g. substitutes $t = 20$ into $\frac{2}{3}r^{\frac{3}{2}} = 72 + \frac{20}{3} \Rightarrow r = \left(\frac{3}{2}\left(72 + \frac{20}{3}\right)\right)^{\frac{2}{3}} = \ldots$

Their work *should* lead to the correct answer so the index work must be correct e.g. $\sqrt{118^3}$ is M0.

$\sqrt[3]{118^2}$ or $118^{\frac{2}{3}}$ are acceptable as values, i.e., the bracket must be evaluated.

May be implied by awrt 24 (cm) following $r^{\frac{3}{2}} = 118$ or by awrt 24.1 (cm). Ignore units for M1.

A1: cao 241mm or 241 or 24.1cm but do not accept 24.1 or e.g. $\sqrt[3]{118^2}$ cm
Correct answer with units implies both marks.

(d)

B1: Examples of acceptable answers (which must relate to the **model in context**):

- (The model will not hold indefinitely as) the balloon may burst/pop
- The balloon is unlikely to be (perfectly) spherical (condone circular)
- The model predicts the balloon will increase in size without limit (which is unrealistic)

Note $t \to \infty \Rightarrow r \to \infty$ is unrealistic / impossible scores B0 unless they reference e.g. the radius.
Condone the presence of additional remarks such as "the balloon may not inflate at the same rate" or "the radius of the balloon might not start at 0" that have already been addressed in the model, but these answers alone score B0.

Question	Scheme	Marks	AOs

12(a)	$\frac{1}{V(25-V)}=\frac{P}{V}+\frac{Q}{25-V}$ e.g. $1=P(25-V)+QV$ $V=0$ or $V=25$ leading to $P=\ldots$ or $Q=\ldots$	M1	1.1b
	$\frac{1}{V(25-V)}=\frac{1}{25V}+\frac{1}{25(25-V)}$	A1	1.1b
		(2)	

Notes

(a)

M1: Sets $\frac{1}{V(25-V)}=\frac{P}{V}+\frac{Q}{25-V}$ and uses a correct method to identify the value of at least one constant.

Do **not** condone incorrect work e.g. $\frac{1}{V(25-V)}=\frac{P}{V}+\frac{Q}{25-V}\Rightarrow 1=PV+Q(25-V)$ etc. this scores M0

A1: Correct partial fractions in any form e.g.

$\frac{1}{25V}+\frac{1}{25(25-V)}$, $\frac{1}{25V}+\frac{1}{625-25V}$, $\frac{1/25}{V}+\frac{1/25}{(25-V)}$, $\frac{1}{25V}-\frac{1}{25(V-25)}$, $\frac{1}{25}\left(\frac{1}{V}+\frac{1}{25-V}\right)$ etc.

Note that this mark is not just for the correct constants, it is for the correctly written fractions either seen in part (a) **or used in part (b).** Allow 0.04 for $\frac{1}{25}$.

Correct partial fractions only scores both marks.

If the correct fractions are obtained following incorrect work score M0A0 but allow full recovery in the rest of the question.

(b) Way 1	$\int \frac{1}{V(25-V)}\,\mathrm{d}V = \int \frac{1}{25V} + \frac{1}{25(25-V)}\,\mathrm{d}V = \frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)$	M1	3.1a
	$\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V) = \frac{1}{10}t(+c)$	A1ft	1.1b
	$t=0, V=20 \Rightarrow \frac{1}{25}\ln 20 - \frac{1}{25}\ln(25-20) = c\left(\Rightarrow c = \frac{1}{25}\ln 4\right)$	M1	3.4
	$V=24 \Rightarrow t = \frac{2}{5}\ln 24 - \frac{2}{5}\ln(25-24) - \frac{2}{5}\ln 4$	**d**M1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
		(5)	
	Alternative for the final 3 marks:		
	$\left[\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\frac{1}{10}t\right]_0^T \Rightarrow \frac{1}{25}\ln 24 - \frac{1}{25}\ln 4 = \frac{1}{10}T$	M1	3.4
	$T = \frac{10}{25}\ln 24 - \frac{10}{25}\ln 4 = \ldots$	**d**M1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
(c)	$\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V) = \frac{1}{10}t + \frac{1}{25}\ln 4$ $\ln V - \ln(25-V) = 2.5t + \ln 4$ $\ln\frac{V}{4(25-V)} = 2.5t \Rightarrow \frac{V}{4(25-V)} = \mathrm{e}^{2.5t}$	M1	2.1
	$\Rightarrow V = 4\mathrm{e}^{2.5t}(25-V) \Rightarrow V + 4V\mathrm{e}^{2.5t} = 100\mathrm{e}^{2.5t} \Rightarrow V = \ldots$	M1	2.1
	$\Rightarrow V = \frac{100\mathrm{e}^{2.5t}}{1+4\mathrm{e}^{2.5t}} = \frac{100}{\mathrm{e}^{-2.5t}+4}$	A1	1.1b
		(3)	
(d)	25 (microlitres)	B1	2.2a
	Since e.g. As $t \to \infty$, $\mathrm{e}^{"-2.5"t} \to 0$	B1	2.4
		(2)	
			(12 marks)

Notes

(b) **Mark (b) and (c) together**

M1: Realises that $\int \frac{\ldots}{V(25-V)}(\mathrm{d}V)$ is required and reaches the form $p\ln\alpha V \pm q\ln\beta(25-V)$ (or e.g. $p\ln\alpha V \pm q\ln\beta(V-25)$) or equivalent for this integration e.g. $p\ln 25V - q\ln(625-25V)$ with p and q non-zero.

But note that $\int \frac{\ldots}{V(25-V)}\mathrm{d}V = \ln V(25-V)$ does not score this mark unless we see an attempt to integrate the partial fractions first.

Condone missing brackets e.g. around the $V-25$ for this mark

Note that the rhs may be incorrect or missing for this mark.

A1ft: Fully correct **<u>equation</u>** following through their P and Q. The "+ c" is **not** required here.

You may need to check carefully when awarding this mark as there will be various alternative correct (or correct ft) forms e.g. these are correct (for correct PF's):

$\frac{1}{25}\ln 25V - \frac{1}{25}\ln(625-25V) = \frac{1}{10}t(+c)$, $\ln V - \ln(25-V) = 2.5t(+c)$,
$\frac{2}{5}\ln 25V - \frac{2}{5}\ln(25-V) = t(+c)$, $\frac{2}{5}\ln 5V - \frac{2}{5}\ln(125-5V) = t(+c)$

In general look for an equation of the form $P\ln\alpha V - Q\ln\beta(25-V) = \frac{1}{10}t(+c)$ or a multiple of this equation. Do **not** condone missing brackets unless they are implied by later work e.g. $\ln 25-V$ for $\ln(25-V)$

Allow $(\ )$ or $|\ |$ around the arguments of the ln's and condone "log" for ln.

M1: States or uses $t = 0$ and $V = 20$ consistently leading to a constant of integration which may be simplified or unsimplified. May be implied by their constant so may need to be checked.
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: States or uses $V = 24$ and proceeds to find a value for t (even if $t < 0$). You do not need to check the processing provided they reach a value for t. **Depends on the previous method mark** and depends on an attempt to integrate both sides however poor.
May be implied by their value for t so may need to be checked.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

Alternative for final 3 marks:

M1: $\left[\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\frac{1}{10}t\right]_0^T \Rightarrow \frac{1}{25}\ln 24 - \frac{1}{25}\ln 4 = \frac{1}{10}T$

Applies the limits 20 and 24 to lhs and 0 to "T" or e.g. "t" on rhs
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: $T = \frac{10}{25}\ln 24 - \frac{10}{25}\ln 4 = ...$ Solves to find "T". You do not need to check the processing provided they reach a value for t/T. **Depends on the previous method mark** and depends on having made some attempt to integrate both sides however poor.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

Note

$$\int \frac{1}{25V} + \frac{1}{25(25-V)}\,dV = \int \frac{1}{25V} - \frac{1}{25V-625}\,dV = \frac{1}{25}\ln V - \frac{1}{25}\ln(25V-625) = \frac{1}{10}t(+c)$$

is also correct integration and scores M1A1

The subsequent marks are also available as described above and could lead to the correct answer if the limits and constant of integration are dealt with correctly.
Use review for any examples like these if you are unsure but generally apply the MS as above.

(c) The marks in (c) depend on having integrated their partial fractions to obtain an equation of the form $\pm...\ln...V \pm ...\ln...(25-V) = \pm kt \pm c, \quad k, c \neq 0$ and … are non-zero constants or equivalent if they have already attempted to eliminate the ln's in (b)

e.g. $\frac{...V}{...(25-V)} = e^{\pm kt \pm c}$ oe

M1: Uses fully correct log work, having obtained a constant of integration, to eliminate all the ln's including from e.g. $e^{\ln 4}$. We condone sign or coefficient slips only.

M1: Proceeds from an **equation of the form** $\frac{...V}{...(25-V)} = ...e^{...t}$ oe using correct algebra to

$V = \ldots$ e.g. $\frac{...V}{...(25-V)} = ...e^{...t} \Rightarrow ...V = ...(25-V)...e^{...t} \Rightarrow (...\pm...)V = ...e^{...t} \Rightarrow V = ...$

Condone sign/coefficient slips only.

A1: Correct expression not just values for the constants.

(d)

B1: Correct value of 25 seen
Allow e.g. < 25 or „ 25
Condone " > 25" but the following mark is then not available

B1: Depends on a correct final equation in any form in (c) e.g. $V = \frac{100e^{2.5t}}{1+4e^{2.5t}}$ oe **and one of**:

- Considers the behaviour as $t \to \infty$ e.g. states that as $t \to \infty$, $e^{-2.5t} \to 0$ (condone "= 0") oe
- $V < 25$ as $\ln(25 - V)$ is not possible when $V \ldots 25$
- Verifies the 25 using a value of t, $t \ldots 9$

Using the differential equation:

B1: Correct value of 25 seen
Allow e.g. " < 25" or „ 25
Condone " > 25" but the following mark is then not available

B1: E.g. when $V = 25$, $\frac{dV}{dt} = 0$ **or** $\frac{dV}{dt} < 0$ if $V > 25$

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
14(a)	$\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \Rightarrow A = ..., B = ...$	M1	1.1b
	Either $A = 2$ or $B = -1$	A1	1.1b
	$\frac{3}{(2x-1)(x+1)} = \frac{2}{2x-1} - \frac{1}{x+1}$	A1	1.1b
		(3)	
(b)	$\int \frac{1}{V} \mathrm{d}V = \int \frac{3}{(2t-1)(t+1)} \mathrm{d}t$	B1	1.1a
	$\int \frac{2}{2t-1} - \frac{1}{t+1} \mathrm{d}t = ...\ln(2t-1) - ...\ln(t+1)(+c)$	M1	3.1a
	$\ln V = \ln(2t-1) - \ln(t+1)(+c)$	A1ft	1.1b
	Substitutes $t = 2, V = 3 \Rightarrow c = (\ln 3)$	M1	3.4
	$\ln V = \ln(2t-1) - \ln(t+1) + \ln 3$ $V = \frac{3(2t-1)}{(t+1)}$ *	A1*	2.1
		(5)	
	(b) Alternative separation of variables:		
	$\int \frac{1}{3V} \mathrm{d}V = \int \frac{1}{(2t-1)(t+1)} \mathrm{d}t$	B1	1.1a
	$\frac{1}{3}\int \frac{2}{2t-1} - \frac{1}{t+1} \mathrm{d}t = ...\ln(2t-1) - ...\ln(t+1)(+c)$	M1	3.1a
	$\frac{1}{3}\ln 3V = \frac{1}{3}\ln(2t-1) - \frac{1}{3}\ln(t+1)(+c)$	A1ft	1.1b
	Substitutes $t = 2, V = 3 \Rightarrow c = \left(\frac{1}{3}\ln 3\right)$	M1	3.4
	$\frac{1}{3}\ln V = \frac{1}{3}\ln(2t-1) - \frac{1}{3}\ln(t+1) + \frac{1}{3}\ln 3$ $V = \frac{3(2t-1)}{(t+1)}$ *	A1*	2.1
		(5)	
(c)	(i) 30 (minutes)	B1	3.2a
	(ii) 6 (m^3)	B1	3.4
		(2)	
			(10 marks)

Notes:

(a)

M1: Correct method of partial fractions leading to values for their A and B

E.g. substitution: $\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \Rightarrow 3 = A(x+1) + B(2x-1) \Rightarrow A = ..., B = ...$

Or compare coefficients $\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \Rightarrow 3 = x(A+2B) + A - B \Rightarrow A = ..., B = ...$

Note that $\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \Rightarrow 3 = A(2x-1) + B(x+1) \Rightarrow A = ..., B = ...$ scores M0

A1: Correct partial fractions not just values for "A" and "B". $\frac{}{2x-1}-\frac{}{x+1}$ or e.g. $\frac{}{2x-1}+\frac{}{x+1}$

Must be seen as **fractions** but if not stated here, allow if the correct fractions appear later.

(b)

B1: Separates variables $\int \frac{1}{V}\mathrm{d}V = \int \frac{3}{(2t-1)(t+1)}\,\mathrm{d}t$. May be implied by later work.

Condone omission of the integral signs but the dV and dt must be in the correct positions if awarding this mark in isolation but they may be implied by subsequent work.

M1: Correct attempt at integration of the partial fractions.

Look for $...\ln(2t-1)+...\ln(t+1)$ where … are constants.

Condone missing brackets around the $(2t – 1)$ and/or the $(t + 1)$ for this mark

A1ft: Fully correct equation following through their A and B **only**.

No requirement for $+c$ here.

The brackets around the $(2t – 1)$ and/or the $(t + 1)$ must be seen or implied for this mark

M1: Attempts to find "c" or e.g. "ln k" using $t = 2,\ V = 3$ following an attempt at integration.

Condone poor algebra as long as $t = 2,\ V = 3$ is used to find a value of their constant.

Note that the constant may be found immediately after integrating or e.g. after the ln's have been combined.

A1*: Correct processing leading to the given answer $V = \frac{3(2t-1)}{(t+1)}$

Alternative:

B1: Separates variables $\int \frac{1}{3V}\mathrm{d}V = \int \frac{1}{(2t-1)(t+1)}\,\mathrm{d}t$. May be implied by later work.

Condone omission of the integral signs but the dV and dt must be in the correct positions if awarding this mark in isolation but they may be implied by subsequent work.

M1: Correct attempt at integration of the partial fractions.

Look for $...\ln(2t-1)+...\ln(t+1)$ where … are constants.

Condone missing brackets around the $(2t – 1)$ and/or the $(t + 1)$ for this mark

A1ft: Fully correct equation following through their A and B **only**.

No requirement for $+c$ here.

The brackets around the $(2t – 1)$ and/or the $(t + 1)$ must be seen or implied for this mark

M1: Attempts to find "c" or e.g. "ln k" using $t = 2,\ V = 3$ following an attempt at integration.

Condone poor algebra as long as $t = 2,\ V = 3$ is used to find a value of their constant.

Note that the constant may be found immediately after integrating or e.g. after the ln's have been combined.

A1*: Correct processing leading to the given answer $V = \frac{3(2t-1)}{(t+1)}$

(Note the working may look like this:

$$\frac{1}{3}\ln 3V = \frac{1}{3}\ln(2t-1) - \frac{1}{3}\ln(t+1) + c,\ \frac{1}{3}\ln 9 = \frac{1}{3}\ln(3) - \frac{1}{3}\ln 3 + c,\ c = \frac{1}{3}\ln 9$$

$$\ln 3V = \ln\frac{9(2t-1)}{(t+1)} \Rightarrow 3V = \frac{9(2t-1)}{(t+1)} \Rightarrow V = \frac{3(2t-1)}{(t+1)}*)$$

Note that B0M1A1M1A1 is not possible in (b) as the B1 must be implied if all the other marks have been awarded.

Note also that some candidates may use different variables in (b) e.g.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y}{(2x-1)(x+1)} \Rightarrow \int \frac{1}{y}\mathrm{d}y = \int \frac{3}{(2x-1)(x+1)}\,\mathrm{d}x$ etc. In such cases you should award marks for equivalent work but they must revert to the given variables at the end to score the final mark.

Also if e.g. a "t" becomes an "x" within their working but is recovered allow full marks.

B1: Deduces 30 minutes. Units not required so just look for 30 but allow equivalents e.g. ½ an hour.
If units are given they must be correct so do not allow e.g. 30 hours.

B1: Deduces 6 m^3. Units not required so just look for 6. Condone $V < 6$ or $V \leq 6$
If units are given they must be correct so do not allow e.g. 6 m.

Question	Scheme	Marks	AOs
14 (a)	Uses the model to state $\frac{\mathrm{d}V}{\mathrm{d}t} = -c$ (for positive constant c)	B1	3.1b
	Uses $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with their $\frac{\mathrm{d}V}{\mathrm{d}t} = -c$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$	M1	2.1
	$-c = 4\pi r^2 \times \frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{c}{4\pi r^2} = -\frac{k}{r^2}$ *	A1*	2.2a
		(3)	
(b)	$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2} \Rightarrow \int r^2 \mathrm{d}r = \int -k \mathrm{d}t$ and integrates with one side "correct"	M1	2.1
	$\frac{r^3}{3} = -kt\left(+\alpha\right)$	A1	1.1b
	Uses $t = 0, r = 40 \Rightarrow \alpha = ...$ $\alpha = \frac{64000}{3}$	M1	1.1b
	Uses $t = 5, r = 20$ & $\alpha = ... \Rightarrow k = ...$	M1	3.4
	$r^3 = 64000 - 11200t$ or exact equivalent	A1	3.3
		(5)	
(c)	Uses the equation of their model and proceeds to a limiting value for t E.g. "$64000 - 11200t$" $... 0 \Rightarrow t ...$	M1	3.4
	For times up to and including $\frac{40}{7}$ seconds	A1ft	3.5b
		(2)	
			(10 marks)
Notes:			

B1: Uses the model to state $\frac{\mathrm{d}V}{\mathrm{d}t} = -c$ (for positive constant c).

Any "letter" is acceptable here including k.

Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = c$ is B0 unless they state that c is a negative constant.

M1: For an attempt to use $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$

Allow for an attempt to use $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = \lambda r^2$ (Any constant is fine)

There is no requirement to use the correct formula for the volume of a sphere for this mark.

A1*: Proceeds to the given answer with an intermediate line equivalent to $\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{c}{4\pi r^2}$

If candidate started with $\frac{\mathrm{d}V}{\mathrm{d}t} = -k$ they must provide a minimal explanation how

$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{4\pi r^2} \rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$. E.g $\frac{1}{4\pi}$ is a constant so replace $\frac{k}{4\pi}$ with k

It is not necessary to use the full formula for the volume of a sphere, eg allow $V = \kappa r^3$ but if it has been quoted it must be correct. So using $V = 4r^3$ can potentially score 2 of the 3 marks.

(b)

M1: For the key step of separating the variables correctly AND integrating one side with at least one index correct. The integral signs do not need to be seen.

A1: Correct integration E.g. $\frac{r^3}{3} = -kt(+\alpha)$ or equivalent. The $+\alpha$ is not required for this mark.

This may be awarded if k has been given a value.

M1: Uses the initial conditions to find a value for the constant of integration α

If a constant of integration is not present, or k has been given a pre defined value, then only the first two marks can be awarded in part (b)

The mark may be awarded if the equation has been adapted incorrectly. E.g. each term cube rooted.

M1: Uses the second set of conditions with their value of α to find k

This may be awarded if the equation has been adapted incorrectly. E.g. each term cube rooted.

A1: Obtains any correct equation for the model.

E.g. $r^3 = 64000 - 11200t$ or exact equivalent such as $\frac{r^3}{3} = \frac{64000}{3} - \frac{11200}{3}t$.

ISW after sight of a correct answer. Condone recurring decimals e.g. $21333.\dot{3}$ for $\frac{64000}{3}$

Do not award if **only the** rounded/truncated decimal equivalents to say $\frac{64000}{3}$ is used.

(c)

M1: Recognises that the model is only valid when $r \geqslant 0$ **and uses this to find *t*.** Condone $r > 0$

Award for an attempt to find the value of t when $r = 0$. See scheme.

It must be from an equation of the form $ar^n = b - ct, \quad a, b, c > 0$ which give + ve values of t.

A1ft: Allow valid for times up to (and including) $\frac{40}{7}$ seconds, 5.71 seconds. Allow $t < \frac{40}{7}$ or $t \leqslant \frac{40}{7}$

There is no requirement for the left hand side of the inequality, 0

States invalid for times greater than $\frac{40}{7}$ seconds, 5.71 seconds.

Follow through on their equation so allow $t <$ their "$\frac{64000}{11200}$" as long as this value is greater than 5

($t = 5$ is one of the values in the question)

Question	Scheme	Marks	AOs

14 (a)	$\{u = 4-\sqrt{h} \Rightarrow\}\ \dfrac{\mathrm{d}u}{\mathrm{d}h} = -\dfrac{1}{2}h^{-\frac{1}{2}}$ or $\dfrac{\mathrm{d}h}{\mathrm{d}u} = -2(4-u)$ or $\dfrac{\mathrm{d}h}{\mathrm{d}u} = -2\sqrt{h}$	B1	1.1b
	$\left\{\displaystyle\int \frac{\mathrm{d}h}{4-\sqrt{h}} = \right\} \displaystyle\int \frac{-2(4-u)}{u}\mathrm{d}u$	M1	2.1
	$= \displaystyle\int\left(-\frac{8}{u}+2\right)\mathrm{d}u$	M1	1.1b
	$= -8\ln u + 2u\ \{+c\}$	M1	1.1b
		A1	1.1b
	$= -8\ln\left\lvert 4-\sqrt{h}\right\rvert + 2(4-\sqrt{h}) + c\ = -8\ln\left\lvert 4-\sqrt{h}\right\rvert - 2\sqrt{h}\ + k$ *	A1*	2.1
		(6)	
(b)	$\left\{\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{t^{0.25}(4-\sqrt{h})}{20} = 0 \Rightarrow\right\}\ 4-\sqrt{h} = 0$	M1	3.4
	Deduces any of $0<h<16,\ 0\le h<16,\ 0<h\le 16,\ 0\le h\le 16,$ $h<16,\ h\le 16$ or all values up to 16	A1	2.2a
		(2)	
(c) Way 1	$\displaystyle\int \frac{1}{(4-\sqrt{h})}\mathrm{d}h\ = \int \frac{1}{20}t^{0.25}\mathrm{d}t$	B1	1.1b
	$-8\ln\left\lvert 4-\sqrt{h}\right\rvert - 2\sqrt{h}\ = \dfrac{1}{25}t^{1.25}\ \{+c\}$	M1	1.1b
		A1	1.1b
	$\{t=0,\ h=1 \Rightarrow\}\ -8\ln(4-1) - 2\sqrt{(1)}\ = \dfrac{1}{25}(0)^{1.25} + c$	M1	3.4
	$\Rightarrow c = -8\ln(3) - 2\ \Rightarrow\ -8\ln\left\lvert 4-\sqrt{h}\right\rvert - 2\sqrt{h}\ = \dfrac{1}{25}t^{1.25} - 8\ln(3) - 2$ $\{h=12 \Rightarrow\}\ -8\ln\left\lvert 4-\sqrt{12}\right\rvert - 2\sqrt{12}\ = \dfrac{1}{25}t^{1.25} - 8\ln(3) - 2$	dM1	3.1a
	$t^{1.25} = 221.2795202... \Rightarrow t = \sqrt[1.25]{221.2795...}$ or $t = (221.2795...)^{0.8}$	M1	1.1b
	$t = 75.154... \Rightarrow t =$ 75.2 (years) (3 sf) or awrt 75.2 (years)	A1	1.1b
	Note: You can recover work for part (c) in part (b)	**(7)**	
(c) Way 2	$\displaystyle\int_1^{12} \frac{20}{(4-\sqrt{h})}\mathrm{d}h\ = \int_0^{T} t^{0.25}\mathrm{d}t$	B1	1.1b
	$\left[20(-8\ln\left\lvert 4-\sqrt{h}\right\rvert - 2\sqrt{h})\ \right]_1^{12} = \left[\ \dfrac{4}{5}t^{1.25}\ \right]_0^{T}$	M1	1.1b
		A1	1.1b
	$20(-8\ln(4-\sqrt{12}) - 2\sqrt{12}) - 20(-8\ln(4-1) - 2\sqrt{1}) = \dfrac{4}{5}T^{1.25} - 0$	M1	3.4
		dM1	3.1a
	$T^{1.25} = 221.2795202... \Rightarrow T = \sqrt[1.25]{221.2795...}$ or $T = (221.2795...)^{0.8}$	M1	1.1b
	$T = 75.154... \Rightarrow T =$ 75.2 (years) (3 sf) or awrt 75.2 (years)	A1	1.1b
	Note: You can recover work for part (c) in part (b)	**(7)**	
			(15 marks)

Notes for Question 14	
(a)	
B1:	See scheme. Allow $\mathrm{d}u = -\frac{1}{2}h^{-\frac{1}{2}}\mathrm{d}h$, $\mathrm{d}h = -2(4-u)\mathrm{d}u$, $\mathrm{d}h = -2\sqrt{h}\,\mathrm{d}u$ o.e.
M1:	Complete method for applying $u = 4-\sqrt{h}$ to $\int \frac{\mathrm{d}h}{4-\sqrt{h}}$ to give an expression of the form $\int \frac{k(4-u)}{u}\mathrm{d}u\ ;\ k \neq 0$
Note:	Condone the omission of an integral sign and/or $\mathrm{d}u$
M1:	Proceeds to obtain an integral of the form $\int\left(\frac{A}{u}+B\right)\{\mathrm{d}u\};\ A, B \neq 0$
M1:	$\int\left(\frac{A}{u}+B\right)\{\mathrm{d}u\} \rightarrow D\ln u + Eu;\ A, B, D, E \neq 0;$ with or without a constant of integration
A1:	$\int\left(-\frac{8}{u}+2\right)\{\mathrm{d}u\} \rightarrow -8\ln u + 2u\ ;$ with or without a constant of integration
A1*:	**dependent on all previous marks** Substitutes $u = 4-\sqrt{h}$ into their integrated result and completes the proof by obtaining the printed result $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h}\ + k.$ Condone the use of brackets instead of the modulus sign.
Note:	They must combine $2(4)$ and their $+c$ correctly to give $+k$
Note:	Going from $-8\ln\left\|4-\sqrt{h}\right\| + 2(4-\sqrt{h}) + c$ to $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h}\ + k,$ with no intermediate working or with no incorrect working is required for the final A1* mark.
Note:	Allow A1* for correctly reaching $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h}\ + c + 8$ and stating $k = c + 8$
Note:	Allow A1* for correctly reaching $-8\ln\left\|4-\sqrt{h}\right\| + 2(4-\sqrt{h}) + k\ =\ -8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h}\ + k$
	Alternative (integration by parts) method for the 2nd M, 3rd M and 1st A mark
	$\left\{\int \frac{-2(4-u)}{u}\mathrm{d}u\ =\ \int \frac{2u-8}{u}\,\mathrm{d}u\ \right\} = (2u-8)\ln u - \int 2\ln u\,\mathrm{d}u\ =\ (2u-8)\ln u - 2(u\ln u - u)\ \{+c\}$
2nd M1:	Proceeds to obtain an integral of the form $(Au+B)\ln u - \int A\ln u\,\{\mathrm{d}u\}\ ;\ A, B \neq 0$
3rd M1:	Integrates to give $D\ln u + Eu;\ D, E \neq 0;$ which can be simplified or un-simplified with or without a constant of integration.
Note:	Give 3rd M1 for $(2u-8)\ln u - 2(u\ln u - u)$ because it is an un-simplified form of $D\ln u + Eu$
1st A1:	Integrates to give $(2u-8)\ln u - 2(u\ln u - u)$ or $-8\ln u + 2u$ o.e. with or without a constant of integration.
(b)	
M1:	Uses the context of the model and has an understanding that the tree keeps growing until $\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \Rightarrow 4-\sqrt{h} = 0.$ Alternatively, they can write $\frac{\mathrm{d}h}{\mathrm{d}t} > 0 \Rightarrow 4-\sqrt{h}\ > 0$
Note:	Accept $h = 16$ or 16 used in their inequality statement for this mark.
A1:	See scheme
Note:	A correct answer can be given M1 A1 from any working.

Notes for Question 14	
(c)	**Way 1**
B1:	Separates the variables correctly. dh and dt should not be in the wrong positions, although this mark can be implied by later working. Condone absence of integral signs.
M1:	Integrates $t^{0.25}$ to give $\lambda t^{1.25}$; $\lambda \neq 0$
A1:	Correct integration. E.g. $-8\ln\left\|4-\sqrt{h}\right\|-2\sqrt{h}=\frac{1}{25}t^{1.25}$ or $20(-8\ln\left\|4-\sqrt{h}\right\|-2\sqrt{h})=\frac{4}{5}t^{1.25}$ $-8\ln\left\|4-\sqrt{h}\right\|+2(4-\sqrt{h})=\frac{1}{25}t^{1.25}$ or $20(-8\ln\left\|4-\sqrt{h}\right\|+2(4-\sqrt{h}))=\frac{4}{5}t^{1.25}$ with or without a constant of integration, e.g. k, c or A
Note:	There is no requirement for modulus signs.
M1:	Some evidence of ***applying*** both $t=0$ and $h=1$ to their model (which can be a changed equation) which contains a constant of integration, e.g. k, c or A
dM1:	**dependent on the previous M mark** Complete process of finding their constant of integration, followed by applying $h=12$ and their constant of integration to their changed equation
M1:	Rearranges their equation to make $t^{\text{their } 1.25}=...$ followed by a correct method to give $t=...;\ t>0$
Note:	$t^{\text{their } 1.25}=...$ can be negative, but their '$t=...$' must be positive
Note:	"their 1.25" cannot be 0 or 1 for this mark
Note:	Do not give this mark if $t^{\text{their } 1.25}=...$ (usually $t^{0.25}=...$) is a result of substituting $t=12$ (or $t=11$) into the given $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{t^{0.25}(4-\sqrt{h})}{20}$. **Note:** They will usually write $\frac{\mathrm{d}h}{\mathrm{d}t}$ as either 12 or 11.
A1:	awrt 75.2
(c)	**Way 2**
B1:	Separates the variables correctly. dh and dt should not be in the wrong positions, although this mark can be implied by later working.
Note:	Integral signs and limits are not required for this mark.
M1:	Same as Way 1 (ignore limits)
A1:	Same as Way 1 (ignore limits)
M1:	Applies limits of 1 and 12 to their model (i.e. to their changed expression in h) and subtracts
dM1	**dependent on the previous M mark** Complete process of applying limits of 1 and 12 and 0 and T (or 't') appropriately to their changed equation
M1:	Same as Way 1
A1:	Same as Way 1

Pearson Education Limited. Registered company number 872828
with its registered office at 80 Strand, London, WC2R 0RL, United Kingdom

Question	Scheme	Marks	AOs
10(a)	$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{H\cos 0.25t}{40}\Rightarrow\int\frac{1}{H}\mathrm{d}H=\int\frac{\cos 0.25t}{40}\,\mathrm{d}t$	M1	3.1a
	$\ln H=\frac{1}{10}\sin 0.25t\,(+c)$	M1 A1	1.1b 1.1b
	Substitutes $t=0, H=5\Rightarrow c=\ln(5)$	dM1	3.4
	$\ln\left(\frac{H}{5}\right)=\frac{1}{10}\sin 0.25t\Rightarrow H=5\mathrm{e}^{0.1\sin 0.25t}$ *	A1*	2.1
		(5)	
(b)	Max height = $5\mathrm{e}^{0.1}=5.53$ m (Condone lack of units)	B1	3.4
		(1)	
(c)	Sets $0.25t=\frac{5\pi}{2}$	M1	3.1b
	31.4	A1	1.1b
		(2)	
			(8 marks)

(a)

M1: Separates the variables to reach $\int\frac{1}{H}\mathrm{d}H=\int\frac{\cos 0.25t}{40}\,\mathrm{d}t$ or equivalent.

The integral signs need to be present on both sides and the $\mathrm{d}H$ AND $\mathrm{d}t$ need to be in the correct positions.

M1: Integrates both sides to reach $\ln H=A\sin 0.25t$ or equivalent with or without the $+c$

A1: $\ln H=\frac{1}{10}\sin 0.25t+c$ or equivalent with or without the $+c$. Allow two constants, one either side

If the 40 was on the lhs look for $40\ln H=4\sin 0.25t+c$ or equivalent.

dM1: Substitutes $t=0, H=5\Rightarrow c=..$ There needs to have been a single "$+c$" to find.

It is dependent upon the previous M mark. You may allow even if you don't explicitly see "$t=0, H=5$" as it may be implied from their previous line

If the candidate has attempted to change the subject and made an error/ slip then condone it for this M but not the final A. Eg. $40\ln H=4\sin 0.25t+c\Rightarrow\Rightarrow H^{40}=\mathrm{e}^{4\sin 0.25t}+\mathrm{e}^{c}\Rightarrow 5^{40}=1+\mathrm{e}^{c}\Rightarrow c=...$

Also many students will be attempting to get to the given answer so condone the method of finding c =... These students will lose the A1* mark

A1*: Proceeds via $\ln H=\frac{1}{10}\sin 0.25t+\ln 5$ or equivalent to the given answer $H=5\mathrm{e}^{0.1\sin 0.25t}$ with at least one correct intermediate line and no incorrect work.

DO NOT condone c's going to c's when they should be e^{c} or A

Accept as a minimum $\ln H = \frac{1}{10}\sin 0.25t + \ln 5 \Rightarrow H = e^{10}$ or $H = e^{10} \times e^{+\ln 5}$ before sight of the given answer

If the only error was to omit the integration signs on line 1, thus losing the first M1, allow the candidate to have access to this mark following a correct intermediate line (see above).

If they attempt to change the subject first then the constant of integration must have been adapted if the A1* is to be awarded. $\ln H = \frac{1}{10}\sin 0.25t + c \Rightarrow H = e^{\frac{1}{10}\sin 0.25t + c} \Rightarrow H = Ae^{\frac{1}{10}\sin 0.25t}$

The dM1 and A1* under this method are awarded at virtually the same time.

Also, for the final two marks, you may see a proof from $\int_0^H \frac{40}{H}\,dH = \int_5^t \cos 0.25t\,dt$

……

There is an alternative via the use of an integrating factor.

……

(b)

B1: States that the maximum height is 5.53 m Accept $5e^{0.1}$ Condone a lack of units here, but penalise if incorrect units are used.

(c)

M1: For identifying that it would reach the maximum height for the 2nd time when $0.25t = \frac{5\pi}{2}$ or 450

A1: Accept awrt 31.4 or 10π Allow if units are seen