adamjee maths

A-LEVEL MATHS / EDEXCEL / FREE RESOURCES

Differentiation

Mark scheme

13 questions

Exam questions and marking instructions remain credited to their board.

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Question	Scheme	Marks	
11 (a)	Uses $t=0, V=20\,000 \Rightarrow 20\,000 = 1500 + A\mathrm{e}^{0}$	M1	3.4
	$A = 18500$	A1	1.1b
	$t=2.5, V=12\,000 \Rightarrow 12\,000 = 1500 + 18\,500\mathrm{e}^{-k\times 2.5}$ $\Rightarrow 10500 = 18500\mathrm{e}^{-k\times 2.5} \Rightarrow k = ...\left(= -\frac{2}{5}\ln\frac{21}{37} = \text{awrt } 0.227\right)$	dM1	3.1b
	$V = 1500 + 18500\,\mathrm{e}^{-0.227t}$	A1	3.3
		(4)	
(b)	Achieves $\left(\frac{\mathrm{d}V}{\mathrm{d}t} =\right) -kA\mathrm{e}^{-kt}$ or $"-0.227"\times"18500"\mathrm{e}^{"-0.227"t}$	B1	1.1b
	Substitutes $A\mathrm{e}^{-kt} = V - 1500$ into $\left(\frac{\mathrm{d}V}{\mathrm{d}t} =\right) -kA\mathrm{e}^{-kt}$ or $"18500"\mathrm{e}^{"-0.227"t} = V - 1500$ into $"-0.227"\times"18500"\mathrm{e}^{"-0.227"t}$	M1	3.4
	Rate of change in value of car is $\left(\frac{\mathrm{d}V}{\mathrm{d}t} =\right) -k(V-1500)$ or $"-0.227"(V-1500)$ *	A1*	2.1
		(3)	
(c)	Suggests a suitable limitation of the model (see notes)	B1	3.5b
		(1)	
			(8 marks)

Notes:

Mark (a) and (b) together

(a)

M1: Uses the equation of the model with $t=0, V=20\,000 \Rightarrow 20000 = 1500 + A\mathrm{e}^{0}$ o.e.
May be implied by 18500

A1: $A = 18500$ (18500 with no working seen scores M1A1) Ignore £ if present.

dM1: Attempts to use the equation of the model $t=2.5, V=12\,000$
$\Rightarrow 12000 = 1500 + "18500"\mathrm{e}^{-2.5k}$ and proceeds to $C\mathrm{e}^{\pm k\times 2.5} = D$ (where $C\times D > 0$ and allow $C=1$) **before proceeding** to find a value for *k*. Allow them to have a non-numerical *C* for this mark.
Note it cannot be implied by their awrt ± 0.227 so
$12000 = 1500 + "18500"\mathrm{e}^{-2.5k} \Rightarrow k = 0.227$ scores dM0A0 as we need to see the intermediate stage $C\mathrm{e}^{\pm k\times 2.5} = D$. (typically look for $18\,500\mathrm{e}^{-k\times 2.5} = 10\,500 \Rightarrow k = ...$ or condone to be implied by a correct expression involving logarithms for their *A*)
It is dependent on the previous method mark.

A1: $V = 1500 + 18500\,\mathrm{e}^{\text{awrt} -0.227t}$ o.e. e.g. $t = \dfrac{\ln\left(\dfrac{V-1500}{18500}\right)}{\text{awrt} -0.227}$ Allow *k* to be exact.

B1: Differentiates to a form $\left(\frac{dV}{dt}=\right)-kAe^{-kt}$ where k and A may be their values from (a)

e.g. $"-0.227"\times"18500"e^{"-0.227"t}$ May just see e.g. (using an exact k) $"-4191.3..."e^{"-0.227"t}$ or e.g. (using k to 3sf) $"-4199.5"e^{"-0.227"t}$ but do not be too concerned by over rounding provided the intention is clear that it is their $-kA$

Do not be too concerned by the left hand side / poor labelling of the derivative.

M1: Substitutes $Ae^{-kt}=V-1500$ into their $\left(\frac{dV}{dt}=\right)\pm kAe^{-kt}$ to form an expression for $\frac{dV}{dt}$ in terms of V. May see e.g. $"-0.227"\times"18500"e^{"-0.227"t}\Rightarrow"-0.227"(V-1500)$.

Condone recovery of a sign slip on the index on $e^{-kt}\rightarrow e^{kt}$ provided it is before they substitute $Ae^{-kt}=V-1500$ for this mark.

Using the given answer: substitutes $V-1500=Ae^{-kt}$ and shows that $-k(V-1500)=-kAe^{-kt}$ (may be in terms of their A and k)

A1*: Full and complete proof with sight of $\frac{dV}{dt}$ oe seen somewhere in their solution and no errors seen. Must see $-kAe^{-kt}$ (or using their values) before proceeding to the given answer which may be written using their numerical value for k

Using the given answer they must conclude that $\frac{dV}{dt}=-k(V-1500)$ which may be written using their numerical value for k

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Alt (b) Separating the variables – may be in terms of their numerical value for *k*

B1ft: Separates the variables correctly and integrates to $\ln(V-1500)=-kt$ with or without $+c$

$$\int\frac{1}{(V-1500)}dV=\int-k\,dt\Rightarrow\ln(V-1500)=-kt\ \ (+c)\ \text{ oe}$$

M1: Proceeds from $...\ln(V-1500)=...kt+c$ o.e. and rearranges to make V the subject. Condone slips.

e.g. $\ln(V-1500)=-kt+c\Rightarrow V-1500=Ae^{-kt}\Rightarrow V=1500+Ae^{-kt}$ or $V=1500+e^{-kt+c}$ (just look for proceeding to $V=\ldots$ for this mark though.) May be in terms of their numerical values for k and A

A1*: Achieves $V=1500+Ae^{-kt}$ with no errors seen and concludes that $\frac{dV}{dt}=-k(V-1500)$

May be in terms of their numerical values for k and A

B1ft: For a correct rearrangement to $t = -\frac{1}{k}\ln\left(\frac{V-1500}{18500}\right)$ ft on their k

M1: Differentiates $...\ln(V-1500)$ to $\frac{...}{V-1500}$ and then finding $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\left(\frac{\mathrm{d}t}{\mathrm{d}V}\right)}$

Typically look for $t = -\frac{1}{k}\ln\left(\frac{V-1500}{18500}\right) \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}V} = -\frac{1}{k}\left(\frac{1}{V-1500}\right)$ o.e. so they may have an unsimplified version of this e.g.

$$t = -\frac{1}{k}\ln\left(\frac{V-1500}{18500}\right) \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}V} = -\frac{1}{k}\left(\frac{\frac{1}{18500}}{\frac{V-1500}{18500}}\right) \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = -k\left(\frac{\frac{V-1500}{18500}}{\frac{1}{18500}}\right)$$

May be in terms of their numerical k. Condone slips.

A1*: Achieves the given answer with no errors and $\frac{\mathrm{d}V}{\mathrm{d}t}$ seen somewhere in their solution

(c) **Note the question asks for a limitation of the model. If there is ambiguity over whether the response is referring to the model then try putting "the model suggests" or "the model" in front of their comment to see if this is a valid limitation.**
Ignore comments which do not contradict a valid limitation.
If values are given then they must be correct and if it is the value of the car then it must have units (£ or pounds)

B1: **Suitable limitations in context referring to the limitation of the model which score B1**
e.g. (the model suggests) the value/price of the car will never go below £1500
e.g. (the model suggests) the rate of decrease of the value of the car is proportionally the same each year
e.g. (the model suggests) after a certain period of time the car will no longer lose value (condone this property of the model that it will tend to a limit)
e.g. (the model) only takes account of age
e.g. (the model) does not take into account damage / alteration to the car/ mileage
e.g. (the model) predicts that the car's value will always go down
e.g. after many years the car may become worthless whereas the model does not allow for this (valid limitation comparing the car to the model)
e.g. the value of the car may go up where as it only decreases according to the model

Do not accept vague/incorrect/irrelevant or non-contextual comments which score B0
e.g. after many years the car may become worthless
e.g. the value of the car may increase
e.g. damage or alterations to the car may impact the value
e.g. the car will still have value when it is very old (should refer to £1500)
e.g. (the model suggests) the minimum value is 1500 (no units for money)
e.g. (the model suggests) the (value of the) car cannot be negative
e.g. car value will fluctuate which the model will not show (a model is not for this purpose)
e.g. the rate of decrease is proportionally the same each year (no context)

Question	Scheme	Marks	AOs
15 (a)	$\frac{1}{2}r^2\theta+\frac{1}{10}r^2=240\Rightarrow r\theta=\frac{240-\frac{1}{10}r^2}{\frac{1}{2}r}$ or $\theta=\frac{240-\frac{1}{10}r^2}{\frac{1}{2}r^2}$	M1 A1	3.4 1.1b
	Substitutes into the expression for P $r\theta=\frac{240-\frac{1}{10}r^2}{\frac{1}{2}r}$ into $\left(P=\right)\ r\theta+2r+\frac{1}{5}r$	dM1	3.4
	$P=\frac{240-\frac{1}{10}r^2}{\frac{1}{2}r}+2r+\frac{1}{5}r=\frac{480}{r}-\frac{1}{5}r+2r+\frac{1}{5}r=2r+\frac{480}{r}$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{\mathrm{d}P}{\mathrm{d}r}=\right)\ 2-\frac{480}{r^2}$	M1	1.1b
	Sets $\frac{\mathrm{d}P}{\mathrm{d}r}=0\Rightarrow r^2=240$ $r=$ awrt 15.5	dM1 A1	2.1 1.1b
		(3)	
(c)	$\left(\frac{\mathrm{d}^2P}{\mathrm{d}r^2}=\right)\ \frac{960}{r^3}$	M1	1.1b
	$\left(\frac{\mathrm{d}^2P}{\mathrm{d}r^2}=\right)$ awrt $0.26>0$ proving a minimum value of P	A1	1.1b
		(2)	
			(9 marks)

Notes:

(a) **Note that just finding a correct equation for the area and/or a correct equation for the perimeter (before any substitution) is insufficient to score any marks.**

M1: Uses area formulae to form an equation of the form $\alpha r^2\theta+\beta r^2=240$ o.e. $(\alpha,\beta\neq 0)$ and rearranges to make $r\theta$, θ or $r\theta+\frac{1}{5}r$ the subject. Look for:

$r\theta=\frac{M\pm Nr^2}{r}\left(=\frac{M}{r}\pm Nr\right)$ o.e. or $\theta=\frac{M\pm Nr^2}{r^2}\left(=\frac{M}{r^2}\pm N\right)$ o.e. where $M,N\neq 0$

or $r\theta+\frac{1}{5}r=\frac{L}{r}$ $L\neq 0$ o.e. May work in degrees.

A1: A correct rearrangement for θ or $r\theta$ or $r\theta+\frac{1}{5}r$ which may be unsimplified (may be in degrees)

$r\theta=\frac{240-\frac{1}{10}r^2}{\frac{1}{2}r}$ o.e. e.g. $r\theta=\frac{2400-r^2}{5r}$ or $r\theta=\frac{480-0.2r^2}{r}$

or $r\theta+\frac{1}{5}r=\frac{480}{r}$ o.e.

or $\theta = \dfrac{240 - \frac{1}{10}r}{\frac{1}{2}r^2}$ o.e. e.g. $\theta = \dfrac{2400 - r^2}{5r^2}$ or $\theta = \dfrac{480}{r^2} - \dfrac{1}{5}$ or $\theta = 480r^{-2} - 0.2$

dM1: Substitutes their $r\theta = \dfrac{M \pm Nr^2}{r}$ o.e. or $\theta = \dfrac{M \pm Nr^2}{r^2}$ o.e. or $r\theta + \dfrac{1}{5}r = \dfrac{L}{r}$ into an expression of the form $(P =)\ r\theta + Qr$, $Q \neq 0$ (typically $P = r\theta + \dfrac{11}{5}r$) which may be unsimplified or in degrees. It is dependent on the previous method mark. It is acceptable for their valid expression for θ, $r\theta$ or $r\theta + \dfrac{1}{5}r$ to be substituted into the perimeter expression directly (without first seeing them in the perimeter expression).

A1*: $P = 2r + \dfrac{480}{r}$ following a correct method (condone slips to be recovered) and all previous marks scored. Condone invisible brackets to be recovered.
$P =$, Perimeter = must be seen at least once in their solution in the correct place.

(b) Mark (b) and (c) together. There is no requirement to see the notation $\dfrac{\mathrm{d}P}{\mathrm{d}r}$ in part (b). It may even be called $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Allow use of e.g. P' or e.g. y'

M1: $\left(\dfrac{\mathrm{d}P}{\mathrm{d}r} =\right)\ p \pm \dfrac{q}{r^2}$ where p and q are non-zero constants

dM1: Sets or implies that their $\dfrac{\mathrm{d}P}{\mathrm{d}r} = 0$ and proceeds to $mr^{\pm 2} = n, \quad m \times n > 0$. It is dependent on the previous method mark. Do not be concerned by the mechanics of the rearrangement. This mark may be implied by a correct answer to their $p - \dfrac{q}{r^2} = 0$. You may need to check this on your calculator.

A1: $r =$ awrt 15.5 or $\sqrt{240}$ $\left(= 4\sqrt{15}\right)$ Do not accept $\pm$ (ignore any units if given)

(c) Condone other letters used instead of *P* and *r* for $\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2}$ e.g. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for M1 only.

Just using $\dfrac{\mathrm{d}P}{\mathrm{d}r}$ and considering a sign change is M0A0

M1: Differentiates and finds $\left(\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2} =\right) \pm \dfrac{f}{r^3}$ (do not be concerned about the sign)

A1: **Note if they score A0 in (b) then this mark cannot be scored.**
Requires

- a correct a correct expression for $\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2}$
- a correct value for $\left(\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2} =\right) \dfrac{960}{r^3} =$ awrt 0.26 using awrt 15.5 (but allow 0.23(43..) if using 16)
- a correct comparison with 0 **and** a conclusion e.g. minimum

The expression for the second derivative does not need to be labelled but if it is then it must be $\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2}$ o.e. or accept e.g. P'' BUT $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ used in their conclusion is A0

1(a)(i)	$y = 4x^3 - 7x^2 + 5x - 10 \Rightarrow \left(\frac{dy}{dx} =\right) 12x^2 - 14x + 5$	M1 A1	1.1b 1.1b
(ii)	$\left(\frac{d^2y}{dx^2} =\right) 24x - 14$	A1ft	1.1b
		(3)	
(b)	$24x - 14 = 0 \Rightarrow x = ...$	M1	1.1b
	$x = \frac{7}{12}$ oe e.g. $x = \frac{14}{24}$	A1	1.1b
		(2)	
			(5 marks)

Notes

(a)(i) If "+ *c*" is included with either derivative penalise it only once on the first occurrence.

M1: Award for $x^3 \to x^2$ or $x^2 \to x$ or $5x \to 5$ or $-10 \to 0$
Indices may be unprocessed e.g. $x^3 \to x^{3-1}$ or $x^2 \to x^{2-1}$ or $5x \to 5x^0$

A1: Correct <u>simplified</u> expression with indices processed $12x^2 - 14x + 5$.
Do **not** allow x^1 for x or $5x^0$ for 5.
Apply isw if necessary once a correct answer is seen.
The "$\frac{dy}{dx} =$" is **not** required.

(ii)

A1ft: Correct simplified second derivative $24x - 14$ or follow through their first derivative.
Must be <u>simplified</u> so do **not** allow e.g. x^1 for x or x^0 for 1 as above.
Apply isw if necessary once a correct answer is seen.
The "$\frac{d^2y}{dx^2} =$" is **not** required.

(b)

M1: Sets their **second derivative** of the form $ax + b$, $a, b \neq 0$ equal to 0 and proceeds to a value for x. Condone slips in rearranging as long as a value for x is obtained.
This may be implied by their value of x or may be implied by their working e.g.
$\left(\frac{d^2y}{dx^2} =\right) 24x - 14 \to 24x = 14 \Rightarrow x = ...$
Condone one slip in copying their second derivative.
Also condone if they "cancel" e.g. $\left(\frac{d^2y}{dx^2} =\right) 24x - 14 \to 12x - 7 = 0 \Rightarrow x = ...$

A1: Correct value from correct work and a correct second derivative but allow recovery if they "cancel" their second derivative to obtain e.g. $12x - 7$.
Allow exact equivalents e.g. $\frac{14}{24}$ but not rounded decimals e.g. 0.583
Allow recurring decimal if clearly indicated e.g. $0.58\dot{3}$
Correct answer only from a correct second derivative (or correctly cancelled second derivative) scores both marks.
Isw after a correct answer is seen.

Question	Scheme	Marks	AOs

4a	$2\alpha+\frac{1}{2}\left(1-\frac{\alpha^2}{2}\right)$	M1	1.2
	$2\alpha+\frac{1}{2}\left(1-\frac{\alpha^2}{2}\right)=0\Rightarrow 2\alpha+\frac{1}{2}-\frac{\alpha^2}{4}=0 \Rightarrow \alpha=...$	dM1	1.1b
	$\alpha=-0.243$ (3dp) only	A1	2.3
		(3)	
b	$f'(0)=\frac{1}{2}\cos 0\Rightarrow ...\Rightarrow y=...x+3$	M1	1.1b
	$y=\frac{1}{2}x+3$	A1	1.1b
		(2)	
			(5 marks)

Notes

(a) **Accept to be in terms of α or another variable e.g. x**

Note: -0.243 **with no working is 0 marks**

M1: Fully substitutes $\cos x=1-\frac{x^2}{2}$ into the derivative.

dM1: Attempts to multiply out to achieve a 3TQ (= 0) **and** attempts to find a value for α. Condone slips. Allow solving the quadratic via any method (usual rules apply).

If they use a calculator then you may need to check this.

A1: $(\alpha=)-0.243$ only cao Can only be scored provided a correct 3TQ is seen. If both roots found then the other one must be rejected (or a choice made of -0.243 e.g. underlining it or a tick)

Condone $x=-0.243$

(b)

M1: Attempts to find the gradient of the curve when $x=0$ and achieves an equation of the form $y="f'(0)"x+3$.

$x=0$ must be fully substituted in and a value must be found for the gradient. Do not allow this mark if they attempt to use a changed gradient e.g. the gradient of the normal.

Also allow attempts using the small angle approximation:

$f'(x)\approx 2x+\frac{1}{2}\left(1-\frac{x^2}{2}\right)$ when $x=0$, $f'(0)="\frac{1}{2}"\Rightarrow y="f'(0)"x+3$

A1: $y=\frac{1}{2}x+3$ or equivalent in the form $y=mx+c$ isw Stating just the values $m=0.5$, $c=3$ without the correct equation is A0

Question	Scheme	Marks	AOs

1(a)	$\{f'(x)=\}\ldots x^2+\ldots x+\ldots \Rightarrow \{f''(x)=\}\ldots x+\ldots$	**M1**	1.1b
	$\{f'(x)=\}3x^2+4x-8 \Rightarrow \{f''(x)=\}6x+4$	**A1cso**	1.1b
		(2)	
(b)(i)	$"6x+4"=0 \Rightarrow x="-\frac{2}{3}"$	**B1ft**	1.1b
(ii)	x „ $"-\frac{2}{3}"$ or $x<"-\frac{2}{3}"$	**B1ft**	2.2a
		(2)	
			(4 marks)

Notes

(a)

M1: For attempting to differentiate twice.

It can be scored for any of: $x^3 \to \ldots x^2 \to \ldots x$ or $2x^2 \to \ldots x \to k$ or $-8x \to k \to 0$ where … are constants.

You can ignore the lhs so do not be concerned what they call the first and/or second derivative, just look for their expressions.

The indices do not need to be processed for this mark so allow for e.g. $x^3 \to \ldots x^{3-1} \to \ldots x^{3-1-1}$

A1cso: $(f''(x)=)\ 6x+4$ Correct second derivative from fully correct work. The "$f''(x)=$" is not required.

Allow $6x^1$ for $6x$ but not $4x^0$ for 4 unless the $4x^0$ becomes 4 later, e.g. in part (b).

Do **not** apply isw so mark their final answer. E.g. if $6x + 4$ becomes $3x + 2$ score A0

(b)

(i)

B1ft: $ax+b=0 \Rightarrow (x=)-\frac{b}{a}$. This mark is for obtaining $x=-\frac{2}{3}$ **or** $x=-\frac{b}{a}$ which has come from solving an equation of the form $ax+b,\ a,b\neq 0$ where $ax + b$ is their attempt to differentiate twice in part (a)

Allow equivalent fractions e.g. $x=-\frac{4}{6}$ or equivalents for their $x=-\frac{b}{a}$ or an exact decimal and isw.

(ii)

B1ft: Deduces x „ $-\frac{2}{3}$ **or** follow through their single value of x from part (i) obtained from their attempt to solve an equation of the form $ax+b=0,\ a,b\neq 0$ where $ax + b$ was their attempt to differentiate twice in part (a) . Do not isw and mark their final answer.

If 2 inequalities are given e.g. $x<"-\frac{2}{3}",\ x>"-\frac{2}{3}"$ without indicating which is their answer score B0

Condone $<$ for „ and allow equivalent inequalities e.g. $-\frac{2}{3}>x$

Allow equivalent fractions e.g. $x=-\frac{4}{6}$ or equivalents for their $x=-\frac{b}{a}$

Allow equivalent notation so these are all acceptable:

x „ $"-\frac{2}{3}",\ x<"-\frac{2}{3}",\ \left(-\infty,"-\frac{2}{3}"\right],\ \left(-\infty,"-\frac{2}{3}"\right),\ \left\{x:x \text{ „ } "-\frac{2}{3}"\right\},\ \left\{x:x<"-\frac{2}{3}"\right\}$

Ignore any reference to values of y.

Allow ft decimal answers from (i) which may be inexact.

Correct answers in part (b) with no working in (a) can score 0011.

11(a)	$\frac{dV}{dh} = 200$ oe e.g. $\frac{dh}{dV} = \frac{1}{200}$	**B1**	1.1b
	$\left(\frac{dh}{dt} =\right) \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}}$	**M1**	3.1a
	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ *	**A1***	2.1
		(3)	
(b)	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \int h^{\frac{1}{2}}\, dh = \int \lambda\, dt \Rightarrow ...h^{\frac{3}{2}} = \lambda t \{+c\}$	**M1**	1.1b
	$\frac{2}{3}h^{\frac{3}{2}} = \lambda t \{+c\}$ oe e.g. $\frac{h^{\frac{3}{2}}}{\frac{3}{2}} = \lambda t \{+c\}$	**A1**	1.1b
	$\frac{2}{3}(1.44)^{\frac{3}{2}} = \lambda \times 0 + c \Rightarrow c = 1.152 \left(= \frac{144}{125}\right)$	**dM1**	3.4
	$\frac{2}{3}(3.24)^{\frac{3}{2}} = \lambda \times 8 + "1.152" \Rightarrow \lambda = 0.342 \left(= \frac{171}{500}\right)$	**ddM1**	3.1b
	$h^{\frac{3}{2}} = 0.513t + 1.728$ oe e.g. $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$	**A1**	3.3
		(5)	
	(b) Alternative:		
	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \frac{dt}{dh} = \frac{\sqrt{h}}{\lambda} \Rightarrow t = ...h^{\frac{3}{2}} (+c)$	**M1**	1.1b
	$t = \frac{2h^{\frac{3}{2}}}{3\lambda}(+c)$ oe	**A1**	1.1b
	$0 = \frac{2(1.44)^{\frac{3}{2}}}{3\lambda} + c$ **and** $8 = \frac{2(3.24)^{\frac{3}{2}}}{3\lambda} + c$ $\Rightarrow \lambda = ...\left(\frac{171}{500}\right)$ **or** $c = ...\left(-\frac{64}{19}\right)$	**dM1**	3.4
	$\Rightarrow \lambda = ...\left(\frac{171}{500}\right)$ **and** $c = ...\left(-\frac{64}{19}\right)$	**ddM1**	3.1b
	$h^{\frac{3}{2}} = 0.513t + 1.728$ oe e.g. $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$	**A1**	3.3
		(5)	
(c)	$5^{\frac{3}{2}} = 0.513t + 1.728 \Rightarrow t = ...$	**M1**	3.4
	$(t =)$ awrt 18.4 min	**A1**	3.2a
		(2)	
			(10 marks)

Notes

(a)

B1: For $\frac{\mathrm{d}V}{\mathrm{d}h}=200$ stated or used – may be implied by their chain rule attempt

M1: Requires:

- $\frac{\mathrm{d}V}{\mathrm{d}h}=p,\quad p>1$
- $\frac{\mathrm{d}V}{\mathrm{d}t}=\pm\frac{k}{\sqrt{h}}$ or e.g. $\frac{\mathrm{d}V}{\mathrm{d}t}=\pm\frac{1}{k\sqrt{h}}$ (or a suitable letter for k, which may be λ, but must **not** be a number)
- **application** of the correct chain rule $\left(\frac{\mathrm{d}h}{\mathrm{d}t}=\right)\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ or any equivalent with $\frac{\mathrm{d}V}{\mathrm{d}t}=\pm\frac{k}{\sqrt{h}}$ or $\pm\frac{1}{k\sqrt{h}}$ and their $\frac{\mathrm{d}V}{\mathrm{d}h}$ correctly placed. So $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{200k}{\sqrt{h}}=\frac{\lambda}{\sqrt{h}}$ scores M0 as $\frac{\mathrm{d}h}{\mathrm{d}V}$ is incorrectly placed.

A1*: A rigorous argument with all steps shown and simplifies to achieve $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\lambda}{\sqrt{h}}$ with no errors.

Do not allow the use of λ for both constants. Allow use of e.g. $\frac{\mathrm{d}V}{\mathrm{d}t}=\pm\frac{1}{k\sqrt{h}}$ for full marks.

e.g. $\frac{\mathrm{d}V}{\mathrm{d}h}=200,\quad\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{200}\times\frac{\lambda}{\sqrt{h}}=\frac{\lambda}{\sqrt{h}}$ scores B1M1A0* *unless* e.g. "let $\lambda=\frac{\lambda}{200}$" seen.

Allow correct work leading to e.g. $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{200}\times\frac{k}{\sqrt{h}}\rightarrow\frac{\lambda}{\sqrt{h}}$ or $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{200}\times\frac{k}{\sqrt{h}}$ so $\lambda=\frac{k}{200}$

There must be an attempt to link the $\frac{\mathrm{d}h}{\mathrm{d}t}$ with the $\frac{\lambda}{\sqrt{h}}$ which may be missing an = sign.

Allow an argument with $\frac{\mathrm{d}V}{\mathrm{d}t}=-\frac{k}{\sqrt{h}}$ e.g. $\frac{\mathrm{d}V}{\mathrm{d}h}=200,\quad\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{200}\times-\frac{k}{\sqrt{h}}=\frac{\lambda}{\sqrt{h}}$

Withhold this A mark if there are notational errors e.g. $\frac{\mathrm{d}V}{\mathrm{d}t}=200,\quad\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{200}\times\frac{k}{\sqrt{h}}=\frac{\lambda}{\sqrt{h}}$

scores B1(implied)M1A0*

(b) Note that some candidates may work with e.g. $\lambda=\frac{k}{200}$ or e.g. $\lambda=200k$ **which is acceptable.**

Candidates who do not have a λ **e.g. assume** $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{200\sqrt{h}}$ or e.g. $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{200}{\sqrt{h}}$ **then only the first 2 method marks are available (see note below). Condone use of other variables if the intention is clear but the final answer must be in terms of *h* and *t*.**

M1: Separates the variables and integrates to obtain an equation of the form $\ldots h^{\frac{3}{2}}=\lambda t\{+c\}$ oe

The constant of integration is not needed for this mark.

A1: $\frac{2}{3}h^{\frac{3}{2}}=\lambda t(+c)$ oe. The constant of integration is not needed for this mark.

Condone spurious notation for this intermediate mark e.g. integral signs left in after integrating.

dM1: Substitutes $t=0$ and $h=1.44$ and attempts to find c.

It is dependent on the previous method mark.

Do not be concerned with the "processing" to find "c" as long as they are using $t=0$ and $h=1.44$

May be implied by their value of c.

ddM1: Substitutes $t=8$ and $h=3.24$ and their c and attempts to find λ. Do not be concerned with the "processing" to find λ as long as they are using $t=8$ and $h=3.24$.

It is dependent on both previous method marks.

A1: Correct equation in the correct form from correct work. $h^{\frac{3}{2}}=0.513t+1.728$ or $h^{\frac{3}{2}}=\frac{513}{1000}t+\frac{216}{125}$

Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Note candidates who do not have a λ e.g. assume $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{200\sqrt{h}}$ or e.g. $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{200}{\sqrt{h}}$ can use either $t=0$ and $h=1.44$ **or** $t=8$ and $h=3.24$ to find their constant of integration.

(b)Alternative:

M1: Finds the reciprocal of both sides and integrates to obtain an equation of the form $t = ...h^{\frac{3}{2}} (+c)$

A1: $t = \frac{2h^{\frac{3}{2}}}{3\lambda}(+c)$ oe. The constant of integration is not needed for this mark.

dM1: Substitutes $t = 0$ and $h = 1.44$ **and** substitutes $t = 8$ and $h = 3.24$ **and** attempts to find λ **or** c.
It is dependent on the previous method mark.
Do not be concerned with the "processing" to find λ or c as long as they are using $t = 0$ and $h = 1.44$ and $t = 8$ and $h = 3.24$ and reach a value for λ or c. May be implied by their value(s).

ddM1: Complete attempt to find λ **and** c. **It is dependent on both previous method marks.**
Do not be concerned with the "processing" to find λ and c as long as they are using $t = 0$ and $h = 1.44$ and $t = 8$ and $h = 3.24$.

A1: Correct equation in the correct form. $h^{\frac{3}{2}} = 0.513t + 1.728$ or $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$
Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Special Case:

Some candidates are using the given equation in part (b) to find the value of A and the value of B using the given conditions. May score a maximum of 00110. This should be marked as follows:

M0A0: (No attempt to integrate)
M1: Substitutes $t = 0$ and $h = 1.44$ to find a value for B
dM1: Substitutes $t = 8$ and $h = 3.24$ with their value of B to find a value for A
A0: Since they have not used the given model.
(Allow full recovery in (c) if this equation is correct)

(c)

M1: Attempts to substitute $h = 5$ into their equation which must be of the form $h^{\frac{3}{2}} = At + B$ or possibly a rearranged equation e.g. $h^{\frac{1}{2}} = \sqrt[3]{At + B}$ with values of A and B leading to a value for t.
Do not be concerned about the processing as long as they use $h = 5$ and obtain a value for t even if t is negative.

A1: Awrt 18.4 minutes **following a correct equation in (b)**.
The units are required but allow e.g. min, mins, but not just 18.4 and not m (which means metres)
Allow e.g. 18 minutes 25 seconds or 18 mins 26 secs
Note this may follow A0 in part (b) as they may have rearranged incorrectly in (b) but use a correct equation in (c) e.g. $\frac{2}{3}h^{\frac{3}{2}} = \frac{171}{500}t + \frac{144}{125}$, $h^{\frac{1}{2}} = \sqrt[3]{\frac{513}{1000}t + \frac{216}{125}}$ or may come from the special case.
Apply isw following a correct time and units, e.g., 18.4 followed by 18 mins.

Question	Scheme	Marks	AOs

14	$y=\dfrac{x-4}{2+\sqrt{x}} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2+\sqrt{x}-\left(x-4\right)\frac{1}{2}x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}$	M1 A1	2.1 1.1b
	$=\dfrac{2+\sqrt{x}-\left(x-4\right)\frac{1}{2}x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}=\dfrac{2+\sqrt{x}-\frac{1}{2}\sqrt{x}+2x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}=\dfrac{2\sqrt{x}+\frac{1}{2}x+2}{\sqrt{x}\left(2+\sqrt{x}\right)^2}$	M1	1.1b
	$=\dfrac{x+4\sqrt{x}+4}{2\sqrt{x}\left(2+\sqrt{x}\right)^2}=\dfrac{\left(2+\sqrt{x}\right)^2}{2\sqrt{x}\left(2+\sqrt{x}\right)^2}=\dfrac{1}{2\sqrt{x}}$	A1	2.1
		(4)	
			(4 marks)

Notes

M1: Attempts to use a correct rule e.g. quotient or product (& chain) rule to achieve the following forms

Quotient : $\dfrac{\alpha\left(2+\sqrt{x}\right)-\beta\left(x-4\right)x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}$ but be tolerant of attempts where the $\left(2+\sqrt{x}\right)^2$ has been incorrectly expanded

Product: $\alpha\left(2+\sqrt{x}\right)^{-1}+\beta x^{-\frac{1}{2}}\left(x-4\right)\left(2+\sqrt{x}\right)^{-2}$

Alternatively with $t=\sqrt{x}$, $y=\dfrac{t^2-4}{2+t} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\times\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{2t\left(2+t\right)-\left(t^2-4\right)}{\left(2+t\right)^2}\times\dfrac{1}{2}x^{-\frac{1}{2}}$ with same rules

A1: Correct derivative in any form. Must be in terms of a single variable (which could be *t*)

M1: Following a correct attempt at differentiation, it is scored for multiplying both numerator and denominator by $\sqrt{x}$ and collecting terms to form a single fraction. It can also be scored from $\dfrac{uv'-vu'}{v^2}$

For the $t=\sqrt{x}$, look for an attempt to simplify $\dfrac{t^2+4t+4}{\left(2+t\right)^2}\times\dfrac{1}{2t}$

A1: Correct expression showing all key steps with no errors or omissions. $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ must be seen at least once

Question	Scheme	Marks	AOs
14	$y=\dfrac{x-4}{2+\sqrt{x}} \Rightarrow y=\dfrac{\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)}{2+\sqrt{x}}=\sqrt{x}-2$	M1 A1	2.1 1.1b
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}$	M1 A1	1.1b 2.1
		(4)	
			(4 marks)

Notes

M1: Attempts to use difference of two squares. Can also be scored using

$t=\sqrt{x} \Rightarrow y=\dfrac{t^2-4}{t+2} \Rightarrow y=\dfrac{\left(t+2\right)\left(t-2\right)}{t+2}$

A1: $y=\sqrt{x}-2$ or $y=t-2$

M1: Attempts to differentiate an expression of the form $y=\sqrt{x}+b$

A1: Correct expression showing all key steps with no errors or omissions. $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ must be seen at least once

Question	Scheme	Marks	AOs
9(a)	$f(x)=4\left(x^2-2\right)e^{-2x}$		
	Differentiates to $e^{-2x}\times 8x+4\left(x^2-2\right)\times -2e^{-2x}$	M1 A1	1.1b 1.1b
	$f'(x)=8e^{-2x}\left\{x-\left(x^2-2\right)\right\}=8\left(2+x-x^2\right)e^{-2x}$ *	A1*	2.1
		(3)	
(b)	States roots of $f'(x)=0$ $x=-1,2$	B1	1.1b
	Substitutes one x value to find a y value	M1	1.1b
	Stationary points are $\left(-1,-4e^2\right)$ and $\left(2,8e^{-4}\right)$	A1	1.1b
		(3)	
(c)	(i) Range $\left[-8e^2,\infty\right)$ o.e. such as $g(x)\geqslant -8e^2$	B1ft	2.5
	(ii) For • Either attempting to find $2f(0)-3=2\times -8-3=(-19)$ and identifying this as the **lower bound** • Or attempting to find $2\times "8e^{-4}"-3$ and identifying this as the **upper bound**	M1	3.1a
	Range $\left[-19,16e^{-4}-3\right]$	A1	1.1b
		(3)	
			(9 marks)
Notes:			

(a)

M1: Attempts the product rule and uses $e^{-2x}\to ke^{-2x}$, $k\neq 0$

If candidate states $u=4(x^2-2), v=e^{-2x}$ with $u'=....,v'=...e^{-2x}$ it can be implied by their $vu'+uv'$

If they just write down an answer without working award for $f'(x)=pxe^{-2x}\pm q\left(x^2-2\right)e^{-2x}$

They may multiply out first $f(x)=4x^2e^{-2x}-8e^{-2x}$. Apply in the same way condoning slips

Alternatively attempts the quotient rule on $f(x)=\dfrac{u}{v}=\dfrac{4\left(x^2-2\right)}{e^{2x}}$ with $v'=ke^{2x}$ and $f'(x)=\dfrac{vu'-uv'}{v^2}$

A1: A correct $f'(x)$ which may be unsimplified.

Via the quotient rule you can award for $f'(x)=\dfrac{8xe^{2x}-8\left(x^2-2\right)e^{2x}}{e^{4x}}$ o.e.

A1*: Proceeds correctly to given answer showing all necessary steps.

The $f'(x)$ or $\dfrac{dy}{dx}$ must be present at some point in the solution

This is a "show that" question and there must not be any errors. All bracketing must be correct.

Allow a candidate to move from the **simplified** unfactorised answer of $f'(x)=8xe^{-2x}-8\left(x^2-2\right)e^{-2x}$

Do not allow it from an **unsimplified** $f'(x) = 4\times 2xe^{-2x} + 4\left(x^2-2\right)\times -2e^{-2x}$

Allow the expression / bracketed expression to be written in a different order.

So, for example, $8\left(x-x^2+2\right)e^{-2x}$ is OK

(b)

B1: States or implies $x = -1, 2$ (as the roots of $f'(x) = 0$)

M1: Substitutes one x value of their solution to $f'(x) = 0$ in f(x) to find a y value.

Allow decimals here (3sf). FYI, to 3 sf, $-4e^2 = -29.6$ and $8e^{-4} = 0.147$

Some candidates just write down the x coordinates but then go on in part (c) to find the ranges using the y coordinates. Allow this mark to be scored from work in part (c)

A1: Obtains $\left(-1,-4e^2\right)$ and $\left(2,8e^{-4}\right)$ as the stationary points. This must be scored in (b). Remember to isw after a correct answer. Allow these to be written separately. E.g. $x=-1,\ y=-4e^2$

Extra solutions, e.g. from $x=0$ will be penalised on this mark.

(c)(i)

B1ft: For a correct range written using correct notation.

Follow through on $2\times$ their minimum "y" value from part (b), providing it is negative.

Condone a decimal answer if this is consistent with their answer in (b) to 3sf or better.

Examples of correct responses are $\left[-8e^2,\infty\right)$, $g\geqslant -8e^2$, $y\geqslant -8e^2$, $\left\{q\in\mathbb{R}, q\geqslant -8e^2\right\}$

(c)(ii)

M1: See main scheme. Follow through on $2\times$ their "$8e^{-4}$" -3 for the upper bound.

A1: Range $\left[-19, 16e^{-4}-3\right]$ o.e. such as $-19\leqslant y\leqslant 16e^{-4}-3$ but must be exact

Question	Scheme	Marks	AOs
15 (a)	$x^2 \tan y = 9 \Rightarrow 2x \tan y + x^2 \sec^2 y \dfrac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\dfrac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$	M1	1.1b
	$\dfrac{dy}{dx} = \dfrac{-2x \times \dfrac{9}{x^2}}{x^2\left(1+\dfrac{81}{x^4}\right)} = \dfrac{-18x}{x^4+81}$ *	A1*	2.1
		(4)	
(b)	$\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$ $\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-18\times\left(x^4+81\right)-\left(-18x\right)\left(4x^3\right)}{\left(x^4+81\right)^2} = \dfrac{54\left(x^4-27\right)}{\left(x^4+81\right)^2}$ o.e.	M1 A1	1.1b 1.1b
	States that when $x < \sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2} < 0$ when $x = \sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2} = 0$ **AND** when $x > \sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2} > 0$ giving a point of inflection when $x = \sqrt[4]{27}$	A1	2.4
		(3)	
			(7 marks)
Notes:			

(a)

M1: Attempts to differentiate $\tan y$ implicitly. Eg. $\tan y \to \sec^2 y \dfrac{dy}{dx}$ or $\cot y \to -\text{cosec}^2 y \dfrac{dy}{dx}$

You may well see an attempt $\tan y = \dfrac{9}{x^2} \Rightarrow \sec^2 y \dfrac{dy}{dx} = ...$

When a candidate writes $x^2 \tan y = 9 \Rightarrow x = 3\tan^{-\frac{1}{2}} y$ the mark is scored for $\tan^{-\frac{1}{2}} y \to ... \tan^{-\frac{3}{2}} y \sec^2 y$

A1: Correct differentiation $2x \tan y + x^2 \sec^2 y \dfrac{dy}{dx} = 0$

Allow also $\sec^2 y \dfrac{dy}{dx} = -\dfrac{18}{x^3}$ or $2x = -9\text{cose c}^2 y \dfrac{dy}{dx}$ amongst others

M1: Full method to get $\dfrac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$

A1*: Proceeds correctly to the given answer of $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

(b)

M1: Attempts to differentiate the given expression using the product or quotient rule.

For example look for a correct attempt at $\frac{vu'-uv'}{v^2}$ with $u=-18x, v=x^4+81, u'=\pm 18, v'=...x^3$

If no method is seen or implied award for $\frac{\pm 18\times\left(x^4+81\right)\pm 18x\left(ax^3\right)}{\left(x^4+81\right)^2}$

Using the product rule award for $\pm 18\left(x^4+81\right)^{-1}\pm 18x\left(x^4+81\right)^{-2}\times cx^3$

A1: Correct **simplified** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{54\left(x^4-27\right)}{\left(x^4+81\right)^2}$ o.e. such as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{54x^4-1458}{\left(x^4+81\right)^2}$

Alternatively score for showing that when a correct (unsimplified) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0\Rightarrow x^4=27\Rightarrow x=\sqrt[4]{27}$

Or for substituting $x=\sqrt[4]{27}$ into an unsimplified but correct $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and showing that it is 0

A1: Correct explanation with a minimal conclusion and correct second derivative.

See scheme.

It can be also be argued from $x^4<27$, $x^4=27$ and $x^4>27$ provided the conclusion states that the point of inflection is at $x=\sqrt[4]{27}$

Alternatively substitutes values of x either side of $\sqrt[4]{27}$ and at $\sqrt[4]{27}$, into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, finds all three values and makes a minimal conclusion.

A different method involves finding $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and showing that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\neq 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ when $x=\sqrt[4]{27}$

FYI $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{23328x^3}{\left(x^4+81\right)^3}=0.219$ when $x=\sqrt[4]{27}$

Alternative part (a) using arctan

M1: Sets $y=\arctan\frac{9}{x^2}\rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times...$ where ... could be 1

A2: $y=\arctan\frac{9}{x^2}\rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times-\frac{18}{x^3}$

A1*: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\frac{81}{x^4}}\times-\frac{18}{x^3}=\frac{-18x}{x^4+1}$ showing correct intermediate step and no errors.

Question	Scheme	Marks	AOs

14 (a)	Attempts to differentiate $x = 4\sin 2y$ and inverts $\frac{dx}{dy} = 8\cos 2y \Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 2y}$	M1	1.1b
	At (0,0) $\frac{dy}{dx} = \frac{1}{8}$	A1	1.1b
		(2)	
(b)	(i) Uses $\sin 2y \approx 2y$ when y is small to obtain $x \approx 8y$	B1	1.1b
	(ii) The value found in (a) is the gradient of the line found in (b)(i)	B1	2.4
		(2)	
(c)	Uses their $\frac{dy}{dx}$ as a function of y and, using both $\sin^2 2y + \cos^2 2y = 1$ and $x = 4\sin 2y$ in an attempt to write $\frac{dy}{dx}$ or $\frac{dx}{dy}$ as a function of x Allow for $\frac{dy}{dx} = k\frac{1}{\cos 2y} = ..\frac{1}{\sqrt{1-(..x)^2}}$	M1	2.1
	A correct answer $\frac{dy}{dx} = \frac{1}{8\sqrt{1-\left(\frac{x}{4}\right)^2}}$ or $\frac{dx}{dy} = 8\sqrt{1-\left(\frac{x}{4}\right)^2}$	A1	1.1b
	and in the correct form $\frac{dy}{dx} = \frac{1}{2\sqrt{16-x^2}}$	A1	1.1b
		(3)	
			(7 marks)

(a)

M1: Attempts to differentiate $x = 4\sin 2y$ and inverts.

Allow for $\frac{dx}{dy} = k\cos 2y \Rightarrow \frac{dy}{dx} = \frac{1}{k\cos 2y}$ or $1 = k\cos 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{k\cos 2y}$

Alternatively, changes the subject and differentiates $x = 4\sin 2y \rightarrow y = ...\arcsin\left(\frac{x}{4}\right) \rightarrow \frac{dy}{dx} = \frac{...}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$

It is possible to approach this from $x = 8\sin y\cos y \Rightarrow \frac{dx}{dy} = \pm 8\sin^2 y \pm 8\cos^2 y$ before inverting

A1: $\frac{dy}{dx} = \frac{1}{8}$ Allow both marks for sight of this answer as long as no incorrect working is seen (See below)

Watch for candidates who reach this answer via $\frac{dx}{dy} = 8\cos 2x \Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 2x}$ This is M0 A0

(b)(i)

B1: Uses $\sin 2y \approx 2y$ when y is small to obtain $x = 8y$ oe such as $x = 4(2y)$.

Do not allow $\sin 2y \approx 2\theta$ to get $x = 8\theta$ but allow recovery in (b)(i) or (b)(ii)

Double angle formula is B0 as it does not satisfy the demands of the question.

(b)(ii)

B1: Explains the relationship between the answers to (a) and (b) (i).

For this to be scored the first three marks, in almost all cases, must have been awarded and the statement must refer to both answers

Allow for example "The gradients are the same $\left(=\frac{1}{8}\right)$" 'both have $m = \frac{1}{8}$'

Do not accept the statement that 8 and $\frac{1}{8}$ are reciprocals of each other unless further correct work explains the relationship in terms of $\frac{\mathrm{d}x}{\mathrm{d}y}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$

(c)

M1: Uses their $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of y and, using both $\sin^2 2y + \cos^2 2y = 1$ and $x = 4\sin 2y$, attempts to write $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}$ as a function of x. The $\frac{\mathrm{d}y}{\mathrm{d}x}$ may not be seen and may be implied by their calculation.

A1: A correct (un-simplified) answer for $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}$ Eg. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8\sqrt{1-\left(\frac{x}{4}\right)^2}}$

A1: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{16-x^2}}$ The $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be seen at least once in part (c) of this solution

.....

Alt to (c) using arcsin

M1: Alternatively, changes the subject and differentiates $x = 4\sin 2y \rightarrow y = ...\arcsin\left(\frac{x}{4}\right) \rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{...}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$

Condone a lack of bracketing on the $\frac{x}{4}$ which may appear as $\frac{x^2}{4}$

A1: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1/8}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$ oe

A1: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{16-x^2}}$

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13 (a)	States or uses $6=\pi r^2 h+\frac{2}{3}\pi r^3$	B1	1.1a
	$\Rightarrow\ h=\frac{6}{\pi r^2}-\frac{2}{3}r,\ \ \pi h=\frac{6}{r^2}-\frac{2}{3}\pi r,\ \ \pi rh=\frac{6}{r}-\frac{2}{3}\pi r^2,\ \ rh=\frac{6}{\pi r}-\frac{2}{3}r^2$		
	$A=\pi r^2+2\pi rh+2\pi r^2\ \{\Rightarrow A=3\pi r^2+2\pi rh\}$		
	$A=2\pi r^2+2\pi r\left(\frac{6}{\pi r^2}-\frac{2}{3}r\right)+\pi r^2$	M1	3.1a
		A1	1.1b
	$A=3\pi r^2+\frac{12}{r}-\frac{4}{3}\pi r^2\ \Rightarrow\ A=\frac{12}{r}+\frac{5}{3}\pi r^2$ *	A1*	2.1
		(4)	
(b)	$\left\{A=12r^{-1}+\frac{5}{3}\pi r^2\Rightarrow\right\}\ \frac{\mathrm{d}A}{\mathrm{d}r}=-12r^{-2}+\frac{10}{3}\pi r$	M1	3.4
		A1	1.1b
	$\left\{\frac{\mathrm{d}A}{\mathrm{d}r}=0\ \Rightarrow\right\}\ -\frac{12}{r^2}+\frac{10}{3}\pi r=0\ \Rightarrow\ -36+10\pi r^3=0\Rightarrow r^{\pm 3}=...\left\{=\frac{18}{5\pi}\right\}$	M1	2.1
	$r=1.046447736...\Rightarrow r=$ 1.05 (m) (3 sf) or awrt 1.05 (m)	A1	1.1b
	Note: Give final A1 for correct exact values for r	**(4)**	
(c)	$A_{\min}=\frac{12}{(1.046...)}+\frac{5}{3}\pi(1.046...)^2$	M1	3.4
	$\{A_{\min}=17.20...\ \Rightarrow\}\ A=17\ (\mathrm{m}^2)$ or $A=$ awrt 17 (m^2)	A1ft	1.1b
		(2)	
			(10 marks)

Notes for Question 13

(a)	
B1:	See scheme
M1:	Complete process of substituting their $h=...$ or $\pi h=...$ or $\pi rh=...$ or $rh=...$, where '...'=f(r) into an expression for the surface area which is of the form $A=\lambda\pi r^2+\mu\pi rh;\ \lambda,\ \mu\neq 0$
A1:	Obtains correct simplified or un-simplified $\{A=\}\ 2\pi r^2+2\pi r\left(\frac{6}{\pi r^2}-\frac{2}{3}r\right)+\pi r^2$
A1*:	Proceeds, using rigorous and careful reasoning, to $A=\frac{12}{r}+\frac{5}{3}\pi r^2$
Note:	Condone the lack of $A=...$ or $S=...$ for any one of the A marks or for both of the A marks
(b)	
M1:	Uses the model (or their model) and differentiates $\frac{\lambda}{r}+\mu r^2$ to give $\alpha r^{-2}+\beta r$; $\lambda,\mu,\alpha,\beta\neq 0$
A1:	$\left\{\frac{\mathrm{d}A}{\mathrm{d}r}=\right\}\ -12r^{-2}+\frac{10}{3}\pi r$ o.e.
M1:	Sets their $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ and rearranges to give $r^{\pm 3}=k,\ k\neq 0$ (**Note:** k can be positive or negative)
Note:	This mark can be implied. Give M1 (and A1) for $-36+10\pi r^3=0\rightarrow r=\left(\frac{18}{5\pi}\right)^{\frac{1}{3}}$ or $r=\left(\frac{36}{10\pi}\right)^{\frac{1}{3}}$ or $r=\left(\frac{3.6}{\pi}\right)^{\frac{1}{3}}$
A1:	$r=$ awrt 1.05 (ignoring units) or $r=$ awrt 105 cm
Note:	Give M0 A0 M0 A0 where $r=$ 1.05 (m) (3 sf) or awrt 1.05 (m) is found from no working.
Note:	Give final A1 for correct exact values for r. E.g. $r=\left(\frac{18}{5\pi}\right)^{\frac{1}{3}}$ or $r=\left(\frac{36}{10\pi}\right)^{\frac{1}{3}}$ or $r=\left(\frac{3.6}{\pi}\right)^{\frac{1}{3}}$

<table>
<tr><th colspan="2">Notes for Question 13 Continued</th></tr>
<tr><td>Note:</td><td>Give final M0 A0 for $-\frac{12}{r^2}+\frac{10}{3}\pi r > 0 \Rightarrow r > 1.0464$</td></tr>
<tr><td>Note:</td><td>Give final M1 A1 for $-\frac{12}{r^2}+\frac{10}{3}\pi r > 0 \Rightarrow r > 1.0464... \Rightarrow r = 1.0464...$</td></tr>
<tr><td>(c)</td><td></td></tr>
<tr><td>M1:</td><td>Substitutes their $r = 1.046...$, found from solving $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ in part (b), into the model with equation $A = \frac{12}{r} + \frac{5}{3}\pi r^2$</td></tr>
<tr><td>Note:</td><td>Give M0 for substituting their r which has been found from solving $\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 0$ or from using $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}$ into the model with equation $A = \frac{12}{r} + \frac{5}{3}\pi r^2$</td></tr>
<tr><td>A1ft:</td><td>$\{A =\}$ 17 or $\{A =\}$ awrt 17 (ignoring units)</td></tr>
<tr><td>Note:</td><td>You can only follow through on values of r for $0.6 \le$ their $r \le 1.3$ (and where their r has been found from solving $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ in part (b))
<table>
<tr><th>r</th><th>A</th><th>A
(nearest integer)</th></tr>
<tr><td>0.6</td><td>21.88495…</td><td>awrt 22</td></tr>
<tr><td>0.7</td><td>19.70849…</td><td>awrt 20</td></tr>
<tr><td>0.8</td><td>18.35103…</td><td>awrt 18</td></tr>
<tr><td>0.9</td><td>17.57448…</td><td>awrt 18</td></tr>
<tr><td>1.0</td><td>17.23598…</td><td>awrt 17</td></tr>
<tr><td>1.1</td><td>17.24463…</td><td>awrt 17</td></tr>
<tr><td>1.2</td><td>17.53982…</td><td>awrt 18</td></tr>
<tr><td>1.3</td><td>18.07958…</td><td>awrt 18</td></tr>
<tr><td>1.05</td><td>17.20124…</td><td>awrt 17</td></tr>
<tr><td>1.04644…</td><td>17.20105…</td><td>awrt 17</td></tr>
</table></td></tr>
<tr><td>Note:</td><td>Give M1 A1 for $A = 17$ (m^2) or $A =$ awrt 17 (m^2) from no working</td></tr>
</table>

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
5	$\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{(2\sin\theta+2\cos\theta)3\cos\theta-3\sin\theta(2\cos\theta-2\sin\theta)}{(2\sin\theta+2\cos\theta)^2}$	M1 A1	1.1b 1.1b
	Expands and uses $\sin^2\theta+\cos^2\theta=1$ at least once in the numerator or the denominator or uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{\ldots}{\ldots\ldots C\sin\theta\cos\theta}$	M1	3.1a
	Expands and uses $\sin^2\theta+\cos^2\theta=1$ the numerator and the denominator AND uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{P}{Q+R\sin 2\theta}$	M1	2.1
	$\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{3}{2+2\sin 2\theta}=\dfrac{\frac{3}{2}}{1+\sin 2\theta}$	A1	1.1b
			(5 marks)

Notes:

M1: For choosing either the quotient, product rule or implicit differentiation and applying it to the given function. Look for the correct form of $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}$ (condone it being stated as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$) but tolerate slips on the coefficients and also condone $\dfrac{\mathrm{d}(\sin\theta)}{\mathrm{d}\theta}=\pm\cos\theta$ and $\dfrac{\mathrm{d}(\cos\theta)}{\mathrm{d}\theta}=\pm\sin\theta$

For quotient rule look for $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{(2\sin\theta+2\cos\theta)\times\pm\ldots\cos\theta-3\sin\theta(\pm\ldots\cos\theta\pm\ldots\sin\theta)}{(2\sin\theta+2\cos\theta)^2}$

For product rule look for

$\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=(2\sin\theta+2\cos\theta)^{-1}\times\pm\ldots\cos\theta\pm3\sin\theta\times(2\sin\theta+2\cos\theta)^{-2}\times(\pm\ldots\cos\theta\pm\ldots\sin\theta)$

Implicit differentiation look for $(\ldots\cos\theta\pm\ldots\sin\theta)y+(2\sin\theta+2\cos\theta)\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\ldots\cos\theta$

A1: A correct expression involving $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}$ condoning it appearing as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

M1: Expands and uses $\sin^2\theta+\cos^2\theta=1$ at least once in the numerator or the denominator OR uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{\ldots}{\ldots\ldots C\sin\theta\cos\theta}$

M1: Expands and uses $\sin^2\theta+\cos^2\theta=1$ in the numerator and the denominator AND uses $2\sin\theta\cos\theta=\sin 2\theta$ in the denominator to reach an expression of the form $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{P}{Q+R\sin 2\theta}$.

A1: Fully correct proof with $A=\dfrac{3}{2}$ stated but allow for example $\dfrac{\frac{3}{2}}{1+\sin 2\theta}$

Allow recovery from missing brackets. Condone notation slips. This is not a given answer

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
9(a)	Either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$	M1	2.1
	$2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$	A1	1.1b
	$(6y-2x)\frac{dy}{dx}=2y-2x$	M1	2.1
	$\frac{dy}{dx}=\frac{2y-2x}{6y-2x}=\frac{y-x}{3y-x}$ *	A1*	1.1b
		(4)	
(b)	$\left(\text{At } P \text{ and } Q \frac{dy}{dx}\to\infty \Rightarrow\right)$ Deduces that $3y-x=0$	M1	2.2a
	Solves $y=\frac{1}{3}x$ and $x^2-2xy+3y^2=50$ simultaneously	M1	3.1a
	$\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$	A1	1.1b
	Using $y=\frac{1}{3}x \Rightarrow x=..$ AND $y=..$	dM1	1.1b
	$P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$	A1	2.2a
		(5)	
(c)	Explains that you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution	B1ft	2.4
		(1)	
			(10 marks)

Notes:

(a)

M1: For selecting the appropriate method of differentiating either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$

It may be quite difficult awarding it for the product rule but condone $-2xy \to -2x\frac{dy}{dx}+2y$ unless you see evidence that they have used the incorrect law *vu'-uv'*

A1: Fully correct derivative $2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

Allow attempts where candidates write $2xdx-2xdy-2ydx+6ydy=0$

but watch for students who write $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}$ This, on its own, is A0 unless you are convinced that this is just their notation. Eg $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

M1: For a valid attempt at making $\frac{dy}{dx}$ the subject. with two terms in $\frac{dy}{dx}$ coming from $3y^2$ and $2xy$

Look for $(...\pm...)\frac{dy}{dx}=.....$ It is implied by $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$

This cannot be scored from attempts such as $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y$ which only has one correct term.

A1*: $\frac{dy}{dx}=\frac{y-x}{3y-x}$ with no errors or omissions.

The previous line $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$ or equivalent must be seen.

(b)

M1: Deduces that $3y-x=0$ oe

M1: Attempts to find either the x or y coordinates of P and Q by solving their $y=\frac{1}{3}x$ with $x^2-2xy+3y^2=50$ simultaneously. Allow for finding a quadratic equation in x or y and solving to find at least one value for x or y.

This may be awarded when candidates make the numerator = 0 ie using $y=x$

A1: $\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$

dM1: Dependent upon the previous M, it is for finding the y coordinate from their x (or vice versa)

This may also be scored following the numerator being set to 0 ie using $y=x$

A1: Deduces that $P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$ OE. Allow to be $x=...$ $y=...$

(c)

B1ft: Explains that this is where $\frac{dy}{dx}=0$ and so you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution (or larger solution).

Allow a follow through for candidates who mix up parts (b) and (c)

Alternatively candidates could complete the square $(x-y)^2+2y^2=50$ and state that y would reach a maximum value when $x=y$ and choose the positive solution from $2y^2=50$

Question	Scheme	Marks	AOs