adamjee maths

A-LEVEL MATHS / EDEXCEL / FREE RESOURCES

Integration

Mark scheme

23 questions

Exam questions and marking instructions remain credited to their board.

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Question	Scheme	Marks	AOs
5 (a)	$\lim_{\delta x \to 0} \sum_{x=1.44}^{2.89} \frac{2}{\sqrt{x}} \delta x = \int_{1.44}^{2.89} \frac{2}{\sqrt{x}} \mathrm{d}x$	B1	1.2
		(1)	
(b)	$= \left[4\sqrt{x}\right]_{1.44}^{2.89} = 4\times1.7 - 4\times1.2$	M1	1.1b
	$= 2$	A1cso	1.1b
		(2)	
			(3 marks)

Notes:

Mark (a) and (b) together

(a)

B1: States that $\int_{1.44}^{2.89} \frac{2}{\sqrt{x}} \mathrm{d}x$ or equivalent such as $2\int_{1.44}^{2.89} x^{-\frac{1}{2}} \mathrm{d}x$ or $2\int_{1.44}^{2.89} x^{-0.5} \mathrm{d}x$ but must include the limits and the dx. Condone $\mathrm{d}x \leftrightarrow \delta x$ as it is very difficult to tell one from another sometimes.

(b)

M1: Uses $\int \frac{1}{\sqrt{x}} \mathrm{d}x = a\sqrt{x}$ or $ax^{\frac{1}{2}}$ (allow a to be 1) **and** applies the given limits to their $ax^{\frac{1}{2}}$ subtracting either way round. (Condone with the constant of integration included)
You do not need to be concerned by fractions within fractions as this is still of the required form e.g. $\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}$. Only condone transcription errors of 2.89 or 1.44 when substituting the limits into the expression.
This mark can be scored for
e.g. $\left[4\sqrt{x}\right]_{1.44}^{2.89} = 4\times\sqrt{2.89} - 4\times\sqrt{1.44}$ or e.g. $\frac{2x^{\frac{1}{2}}}{\frac{1}{2}} \rightarrow \frac{2(2.89)^{\frac{1}{2}} - 2(1.44)^{\frac{1}{2}}}{\frac{1}{2}}$
May already be partially evaluated so allow e.g. $\frac{34}{5} - \frac{24}{5}$ o.e. provided it is not just 2.

A1: 2 cso
The method mark must have been awarded. Do not withhold this mark for poor notation or e.g. a missing dx in their solution.

Question	Scheme	Marks	AOs
13	$\int \frac{x}{(2x+1)^3}\,dx = \frac{x(2x+1)^{-2}}{-4} + \int \frac{(2x+1)^{-2}}{4}(dx)$	M1	3.1a
	$= \ldots + \frac{(2x+1)^{-1}}{-8}$	dM1	1.1b
	$-\frac{x}{4(2x+1)^2} - \frac{1}{8(2x+1)}$	A1	1.1b
	e.g. $= \left(-\frac{2}{4(2\times2+1)^2} - \frac{1}{8(2\times2+1)}\right) - \left(-\frac{0}{4(2\times0+1)^2} - \frac{1}{8(2\times0+1)}\right)$	ddM1	1.1b
	e.g. $-\frac{2}{100} - \frac{1}{40} + \frac{1}{8} = \frac{2}{25}$ *	A1*	2.1
		(5)	
13 Alt I	$\int \frac{x}{(2x+1)^3}\,dx = \int \frac{u-1}{4u^3}(du)$ where $u = 2x+1$	M1	3.1a
	$= \int \frac{1}{4}u^{-2} - \frac{1}{4}u^{-3}\,du = \ldots$	dM1	1.1b
	$= -\frac{1}{4}u^{-1} + \frac{1}{8}u^{-2}$	A1	1.1b
	$\int_0^2 \frac{x}{(2x+1)^3}\,dx = \left[-\frac{1}{4}u^{-1} + \frac{1}{8}u^{-2}\right]_1^5 = \left(-\frac{1}{20} + \frac{1}{200}\right) - \left(-\frac{1}{4} + \frac{1}{8}\right)$	ddM1	1.1b
	e.g. $-\frac{1}{20} + \frac{1}{200} + \frac{1}{8} = \frac{2}{25}$ *	A1*	2.1
		(5)	
			(5 marks)

Notes:

M1: Obtains $\alpha x(2x+1)^{-2} \pm \beta\int(2x+1)^{-2}\,(dx)$ o.e. where $\alpha, \beta \neq 0$ but may be equal to each other (you do not need to be concerned about how they arrive at this)

dM1: Uses a correct method to integrate an expression of the form $\pm\beta\int(2x+1)^{-2}\,(dx) \to \pm\gamma(2x+1)^{-1}, \quad \beta, \gamma \neq 0$

It is dependent on the previous method mark.

A1: $-\frac{x}{4(2x+1)^2} - \frac{1}{8(2x+1)}$ o.e. Allow this to be unsimplified

Watch out for the DI method

	D	I
+	x	$(2x+1)^{-3}$
−	1	$-\frac{(2x+1)^{-2}}{4}$
+	0	$\frac{(2x+1)^{-1}}{8}$

Giving correct integration e.g. $\int \frac{x}{(2x+1)^3}\,\mathrm{d}x = -\frac{x(2x+1)}{4} - \frac{(2x+1)}{8}$

Score M1dM1 for obtaining $\pm px(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ and A1 for both correct.

ddM1: Substitutes 0 and 2 into an expression of the form $\pm px(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ or equivalent and subtracts either way round. Evidence of limits used cannot be the given answer. Condone slips but not directly evaluating the expression as 0 when x is 0. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous marks scored. Condone missing/poor notation e.g. missing the dx throughout

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Alternative method I – substitution using $u = 2x+1$

M1: Uses a suitable substitution e.g. $u = 2x+1$ and proceeds to $A\int \frac{u-1}{u^3}\,(\mathrm{d}u)$ o.e.

dM1: Splits into separate fractions and attempts to integrate $A\int u^{-2} - u^{-3}\,(\mathrm{d}u)$ Look for at least one correct index for one of the two terms i.e. $u^{-2} \to u^{-1}$ or $u^{-3} \to u^{-2}$. It is dependent on the previous method mark.

A1: $-\frac{1}{4}u^{-1} + \frac{1}{8}u^{-2}$ o.e.

ddM1: Substitutes correct limits (1 and 5 if in terms of u) into an expression of the correct form $...u^{-1} + ...u^{-2}$ (or may have substituted back in terms of x and substitutes in 0 and 2 into an expression of the correct form $\pm p(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ o.e and subtracts either way round – see above for ddM1). Evidence of limits used cannot be the given answer. Condone slips. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous marks scored. Condone missing/poor notation e.g. missing the du throughout

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Alternative method II – partial fractions

M1: Writes $\frac{x}{(2x+1)^3}$ as $\frac{0.5}{(2x+1)^2} - \frac{0.5}{(2x+1)^3}$ o.e. Allow $\pm\frac{M}{(2x+1)^2} \pm \frac{N}{(2x+1)^3}$ (where M and N are constants)

dM1: $\int \frac{x}{(2x+1)^3}\,(\mathrm{d}x) = \int \frac{"0.5"}{(2x+1)^2}\,(\mathrm{d}x) - \int \frac{"0.5"}{(2x+1)^3}\,(\mathrm{d}x) = \pm...(2x+1)^{-1} \pm ...(2x+1)^{-2}$

A1: $-\frac{1}{4(2x+1)^1} + \frac{1}{8(2x+1)^2}$ o.e.

ddM1: Substitutes 0 and 2 into an expression of the correct form $\pm...(2x+1)^{-1} \pm ...(2x+1)^{-2}$ and subtracts either way round. Evidence of limits used cannot be the given answer. Condone slips but not directly evaluating the expression as 0 when x is 0. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous

Question	Scheme	Marks	AOs
16 (a)	$R=\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x$		
	$\int \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x=\int \frac{1}{4\sin^2 u\sqrt{4-4\sin^2 u}}2\cos u\,(\mathrm{d}u)$	M1	3.1a
	Uses $1-\sin^2 u=\cos^2 u \quad \Rightarrow \sqrt{4-4\sin^2 u}=2\cos u$	dM1	1.1b
	$=\int \frac{1}{4\sin^2 u\times 2\cos u}2\cos u\,(\mathrm{d}u)=\int \frac{1}{4}\operatorname{cosec}^2 u\,\mathrm{d}u$	A1	2.1
	Deduces $a=\frac{\pi}{6}$ and $b=\frac{\pi}{3}$	B1	2.2a
		(4)	
(b)	$\int \frac{1}{4}\operatorname{cosec}^2 u\,(\mathrm{d}u)=-\frac{1}{4}\cot u \ (+c)$	B1ft	1.2
	$=\left[-\frac{1}{4}\cot u\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=-\frac{1}{4}\cot\frac{\pi}{3}+\frac{1}{4}\cot\frac{\pi}{6}$	M1	2.1
	$=\frac{1}{2\sqrt{3}}$ or $\frac{\sqrt{3}}{6}$	A1	1.1b
		(3)	
			(7 marks)

Notes:

(a)

M1: Proceeds to $\frac{1}{A\sin^2 u\sqrt{\pm C\pm C\sin^2 u}}\times B\cos u$ o.e. or may be implied by $\frac{\cos u}{D\sin^2 u\cos u}$ o.e

in terms of u only $A, B, C, D \neq 0$

Requires $\mathrm{d}x\rightarrow ...\cos u\ (\mathrm{d}u)$ o.e. Condone the omission of the integral sign and du for this mark. Condone the $\pm B\cos u$ appearing after the du if present

It cannot be implied by $\frac{1}{D\sin^2 u}$ or $\frac{\operatorname{cosec}^2 u}{D}$ where $D\neq 0$

dM1: Attempts to use $\pm 1\pm\sin^2 u=\pm\cos^2 u$ to change $\sqrt{\pm C\pm C\sin^2 u}$ to $p\cos u$ o.e. (Must be seen somewhere in their solution which may be part of side-workings)

May be seen as e.g. $\frac{1}{A\sin^2 u\sqrt{C\cos^2 u}}\times B\cos u$ or $\frac{1}{A\sin^2 u\times F\cos u}\times B\cos u$

It is dependent on the previous method mark.

A1: $\int \frac{1}{4}\operatorname{cosec}^2 u\,\mathrm{d}u$ Ignore any limits or the absence of them for this mark.

Requires the integral sign and du. Do not isw. Must be seen in (a)

B1: Deduces $a=\frac{\pi}{6}$ and $b=\frac{\pi}{3}$ (in radians) May be with their final integral in the correct positions or stated separately which value is a and which value is b. Must be seen in (a).

Alternative part (a) (Further Maths)

M1: $u = \arcsin\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{M}{\sqrt{1-Nx^2}}$ It cannot be implied by $\frac{1}{D\sin^2 u}$ or $\frac{\mathrm{cosec}^2 u}{D}$, $D \neq 0$

dM1: Uses $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{4-x^2}}$ o.e , substitutes $x = 2\sin u$ and proceeds to $\frac{1}{4\sin^2 u}$ It is dependent on the previous method mark.

A1: $\int \frac{1}{4}\mathrm{cosec}^2 u \, \mathrm{d}u$ Ignore any limits or the absence of them for this mark.

Requires the integral sign and du. Do not isw. Must be seen in (a)

B1: Deduces $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$ (must be in radians)

May be with their final integral in the correct positions or stated separately which value is a and which value is b. Must be seen in (a).

(b) Condone x instead of u provided the appropriate limits are substituted into the function

B1ft: $k\mathrm{cosec}^2 u \rightarrow -k\cot u$ (may be left in terms of $k \neq 0$)

e.g. $-2\mathrm{cosec}^2 u \rightarrow 2\cot u$ is B1 (look for the sign to change as well)

Note some candidates may prefer to change

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4}\mathrm{cosec}^2 u \left(\mathrm{d}u\right) = -\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{4}\mathrm{cosec}^2 u \left(\mathrm{d}u\right) = \frac{1}{4}\cot u \ (+c) \text{ which scores B1ft}$$

M1: Uses the limits $\frac{\pi}{6}$ and $\frac{\pi}{3}$ either way round (or condone use of $30°$ and $60°$) in an expression of the form $\pm q\cot u$ and subtracts (either way round).

Allow $q = 1$. May be implied by their final answer provided B1ft has been scored.

May write as e.g. $-\frac{1}{4}\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} + \frac{1}{4}\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$

Do not condone a sign slip here e.g. $-\frac{1}{4}\cot\frac{\pi}{3} - \frac{1}{4}\cot\frac{\pi}{6}$ is M0.

The expression is sufficient but if just a value is stated following integration to the required form then you may need to check this on your calculator.

If no algebraic integration is seen then M0

If only decimal values are seen then M0

A1: $\frac{1}{2\sqrt{3}}$ or $\frac{\sqrt{3}}{6}$ provided the previous two marks have been scored.

Note that incorrect integration e.g. $\int k\,\mathrm{cosec}^2 u \, \mathrm{d}u = k\cot u \rightarrow \frac{1}{2\sqrt{3}}$ scores B0ftM1A0

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Question	Scheme	Marks	AOs
2	$x^4 \to \ldots x^5$ or $x^{\frac{1}{2}} \to \ldots x^{\frac{3}{2}}$ or $-3 \to \ldots x$	M1	1.1a
	Any of $\frac{1}{5}x^5$ or $-\frac{6}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}}$ or $-3x$	A1	1.1b
	Any two of $\frac{1}{5}x^5$ or $-\frac{6}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}}$ or $-3x$	A1	1.1b
	$\frac{1}{5}x^5 - 4x^{\frac{3}{2}} - 3x + c$	A1	1.1b
		(4)	
			(4 marks)

Notes

M1: For increasing any power by one. Score for $x^n \to x^{n+1}$ in any term, including, $-3 \to \ldots x$ where … is a constant, but not for + *c*. Allow the indices to be unprocessed, e.g., x^{4+1}

A1: One correct term which may be unsimplified and indices may be unprocessed.

Condone e.g. $-3x^1$ or $-\frac{6}{\left(\frac{1}{2}+1\right)}x^{\frac{1}{2}+1}$ for this mark. Not scored for + *c*

A1: Two correct terms which may be unsimplified but indices must be processed.

Condone $-3x^1$ for this mark. Not scored for + *c*

A1: cao Requires all terms simplified and + *c*

Ignore the LHS i.e. ignore what they call their integral.

Allow $0.2x^5$ for $\frac{1}{5}x^5$ and $-4\sqrt{x^3}$ or $-4\sqrt{x}^3$ or $-4x\sqrt{x}$ or $-4x^{1.5}$ for $-4x^{\frac{3}{2}}$

Do not allow $-3x^1$ for this mark.

Condone spurious integral signs e.g. $\int \frac{1}{5}x^5 - 4x^{\frac{3}{2}} - 3x + c$ or d*x* left in their answer

ISW after a correct expression seen e.g. if they multiply through by 5 or e.g. try to solve = 0

Do not allow e.g. $\frac{1}{5}x^5 + -4x^{\frac{3}{2}} + -3x + c$

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
15(a)	$\left(f'(x)=\right)\ \dfrac{\lambda x\left(1+x^2\right)^2-\mu x\left(1-x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)^4}$	M1	1.1b
	$=\dfrac{-2x\left(1+x^2\right)^2-4x\left(1-x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)^4}$	A1	1.1b
	e.g. $\dfrac{-2x\left(1+x^2\right)-4x\left(1-x^2\right)}{\left(1+x^2\right)^3}=0$ $\Rightarrow\left(1+x^2\right)+2\left(1-x^2\right)=0$ $\Rightarrow 3-x^2=0$ or e.g. $=\dfrac{-2x\left(1+x^2\right)\left[\left(1+x^2\right)+2\left(1-x^2\right)\right]}{\left(1+x^2\right)^4}$ $=\dfrac{-2x\left(1+x^2\right)\left(3-x^2\right)}{\left(1+x^2\right)^4}$	dM1	3.1a
	e.g. $3-x^2=0$ or $2x\left(1+x^2\right)\left(x^2-3\right)=0$ $\Rightarrow x=...\Rightarrow y=\ldots$	ddM1	2.1
	$\left(-\sqrt{3},-\dfrac{1}{8}\right)$	A1	2.3
		(5)	

Notes

(a) There may be other valid ways to differentiate or to solve the resulting equation.

M1: Attempts the quotient rule to achieve $\dfrac{\pm\lambda x\left(1+x^2\right)^2\pm\mu x\left(1-x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)^4}$

but condone $\dfrac{\pm\lambda x\left(1+x^2\right)^2\pm\mu x\left(1-x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)^2}$ provided an incorrect quotient rule is not seen.

Alternatively, attempts the product rule on $f(x)=\left(1-x^2\right)\left(1+x^2\right)^{-2}$ to achieve

$\pm\lambda x\left(1+x^2\right)^{-2}\pm\mu x\left(1-x^2\right)\left(1+x^2\right)^{-3}$

There may be attempts at splitting the numerator to $f(x)=\left(1+x^2\right)^{-2}-x^2\left(1+x^2\right)^{-2}$ which should differentiate to $\pm\lambda x\left(1+x^2\right)^{-3}\pm\mu x\left(1+x^2\right)^{-2}\pm\gamma x^3\left(1+x^2\right)^{-3}$

In **all** cases, do not penalise signs of λ, μ and/or γ unless an incorrect quotient rule or an incorrect product rule is stated.

Any occurrences of $\left(1+x^2\right)^2$ may be replaced with $1+2x^2+x^4$

There is no need for a LHS e.g. $f'(x)=$ to be present so ignore an incorrect LHS.

Invisible brackets may be recovered/implied by later work.

A1: Correct differentiation. May be unsimplified. Ignore absence of (or incorrect) LHS.

The correct derivative using the quotient rule is $=\dfrac{-2x\left(1+x^2\right)^2-4x\left(1-x^2\right)\left(1+x^2\right)}{\left(1+x^2\right)^4}$ o.e.

using the product rule is $-2x\left(1+x^2\right)^{-2}-4x\left(1-x^2\right)\left(1+x^2\right)^{-3}$ o.e.

and splitting the numerator is $-4x\left(1+x^2\right)^{-3}-2x\left(1+x^2\right)^{-2}+4x^3\left(1+x^2\right)^{-3}$ o.e.

You may need to check carefully for equivalent derivatives and ISW after a correct expression is seen. Note that some candidates may set = 0 at the start and may omit the denominator as a result – send to review if you are unsure in such cases.

dM1: Attempts to reduce the expression to a suitable form so that the roots can be found, either by cancelling a factor of $\left(1+x^2\right)$ and simplifying the other brackets

e.g. $\dfrac{-2x\left(1+x^2\right)-4x\left(1-x^2\right)}{\left(1+x^2\right)^3} \rightarrow$ e.g. $\dfrac{-6x+2x^3}{\left(1+x^2\right)^3}$ or $\dfrac{2x\left(x^2-3\right)}{\left(1+x^2\right)^3}$ or $2x\left(x^2-3\right)\ (=0)$

or by attempting to take x, $\left(1+x^2\right)$ or $\pm 2x\left(1+x^2\right)$ out as a factor and simplify the other brackets. If they only take out a factor of x then they must have $x\left(\pm Ax^4 \pm Bx^2 \pm C\right)$

e.g. $\dfrac{-2x\left(1+x^2\right)\left[\left(1+x^2\right)+2\left(1-x^2\right)\right]}{\left(1+x^2\right)^4} \rightarrow$ e.g. $\dfrac{-2x\left(1+x^2\right)\left(3-x^2\right)}{\left(1+x^2\right)^4}$ or $\dfrac{x\left(2x^4-4x^2-6\right)}{\left(1+x^2\right)^4}$

Depends on the first M mark.

Note: they might attempt to expand the brackets first, or set = 0 and multiply through by $\left(1+x^2\right)^k$ and/or divide through by x or $2x$ at any stage.

Do not be concerned if any factors of $\left(1+x^2\right)$, x or 2 go "missing" during their work.

The = 0 does not explicitly need to be seen.

Look for $\pm\dfrac{2x\left(1+x^2\right)\left(Ax^2+B\right)}{\left(1+x^2\right)^k}$ or $\pm\dfrac{\left(1+x^2\right)\left(Ax^3+Bx\right)}{\left(1+x^2\right)^k}$ or $\pm\dfrac{x\left(Ax^4+Bx^2+C\right)}{\left(1+x^2\right)^k}$ where

- k may be 0, 1, 2, 3 or 4
- A and B (and C if present) are non-zero
- any of $2x$ or $\left(1+x^2\right)$ in the numerator and/or $\left(1+x^2\right)^k$ in the denominator may be absent

ddM1: Attempts to solve their numerator set = 0 using a valid non-calculator method **and** uses a solution for x to find a corresponding value for y.

The substitution may be implied by their value of y but, if not, the substitution must be seen.

- For either Ax^2+B or Ax^3+Bx, we require $A \neq B$ and $A \times B < 0$ From either expression they can write down their $x=\pm\sqrt{\dfrac{-B}{A}}$ without working.
- For Ax^4+Bx^2+C they must show a valid non-calculator method for solving the quartic, treating it as a quadratic in x^2 and using the usual rules for solving a quadratic algebraically, e.g. $a=x^2 \rightarrow 2a^2-4a-6\,(=0) \rightarrow (a+1)(2a-6)\,(=0) \rightarrow x=\pm\sqrt{3}$. They must reach a value for x (and not just x^2) as well as finding a value for y.

Dependent on both previous method marks. They must use a value for x that is not –1, 0 or 1.

A1: Deduces the correct **exact** coordinates for $P\left(-\sqrt{3}, -\dfrac{1}{8}\right)$ o.e. $\left(-\sqrt{3}, -0.125\right)$

Requires all the previous marks to have been scored.

Condone $x=-\sqrt{3}$, $y=-0.125$ or e.g. $-\sqrt{3}, -\dfrac{1}{8}$

If there is more than one pair of coordinates given, then the correct coordinates must be clearly selected or any others clearly rejected.

Ignore any mistakes that occur if they multiply out the denominator $\left(1+x^2\right)^4$ after differentiation.

Question	Scheme	Marks	AOs
15(b)	$x=-1\Rightarrow\alpha=-\dfrac{\pi}{4}$ and $x=1\Rightarrow\beta=\dfrac{\pi}{4}$	B1	2.2a
	$\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$	B1	1.1b
	$\displaystyle\int\frac{1-x^2}{\left(1+x^2\right)^2}\mathrm{d}x=\int\frac{1-\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\text{"}\sec^2\theta\text{"}\,\mathrm{d}\theta$ or $=\displaystyle\int\frac{1-\tan^2\theta}{\sec^4\theta}\text{"}\sec^2\theta\text{"}\,\mathrm{d}\theta$	M1	1.1b
	$=\int\left(1-\tan^2\theta\right)\cos^2\theta\,\mathrm{d}\theta$ $=\displaystyle\int\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta\,\mathrm{d}\theta$ $=\int\left(\cos^2\theta-\sin^2\theta\right)\mathrm{d}\theta$ $=\displaystyle\int\frac{2-1-\tan^2\theta}{\sec^2\theta}\mathrm{d}\theta$ $=\displaystyle\int\left(\frac{2}{\sec^2\theta}-\frac{1+\tan^2\theta}{\sec^2\theta}\right)\mathrm{d}\theta$ $=\int\left(2\cos^2\theta-1\right)\mathrm{d}\theta$	dM1	3.1a
	$=\int\cos 2\theta\,\mathrm{d}\theta$ *	A1*	2.1
		(5)	

Notes

(b) **Mark parts (b) and (c) together.**

B1: Deduces the correct limits for the integral in θ which may be seen separately as side working or within their integral work. This mark cannot be scored working in degrees.

Allow $\dfrac{3\pi}{4}$ instead of $-\dfrac{\pi}{4}$ but not decimal approximations.

B1: $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$ or equivalent e.g. $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=1+x^2$ or $\dfrac{\cos^2\theta--\sin^2\theta}{\cos^2\theta}$ (coming from $x=\dfrac{\sin\theta}{\cos\theta}$) but must be correct, so not e.g. $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2 x$ unless recovered.

M1: Makes a complete attempt at using the substitution $x=\tan\theta$

Requires:

- An attempt at using $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ to replace dx with $\mathrm{d}\theta$ either way round, so if $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\mathrm{g}(\theta)$ allow either $\mathrm{d}x=\mathrm{g}(\theta)\{\mathrm{d}\theta\}$ **or** $\mathrm{d}x=\dfrac{1}{\mathrm{g}(\theta)}\{\mathrm{d}\theta\}$. $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ must be a function of θ and not a constant.
- All terms in x replaced with $\tan\theta$ (or $1+x^2$ with $\sec^2\theta$)

Condone the absence of $\mathrm{d}\theta$ but dx must no longer be present.

Condone if they fail to square the denominator or e.g. a slip in missing a θ

Use of notation such as $\mathrm{d}(\tan\theta)$ is correct but does not score the M1 until replaced with $\sec^2\theta\{\mathrm{d}\theta\}$ or their derivative of $\tan\theta$ (which must not be a constant multiple of $\tan\theta$).

dM1: Uses trigonometric identities e.g. $\pm1\pm\tan^2\theta=\pm\sec^2\theta$, $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\sec\theta=\dfrac{1}{\cos\theta}$, $\pm\cos^2\theta\pm\sin^2\theta=\pm1$ to simplify the integral to an expression of the form $\pm a\sin^2\theta\pm b\cos^2\theta\pm c$ where one of a, b or c may be 0.

The algebra should essentially be correct but condone e.g. sign slips or errors collecting terms.

Alternatively, e.g. $\int \frac{1-\tan^2\theta}{1+\tan^2\theta}\mathrm{d}\theta = \int \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\mathrm{d}\theta = \int \frac{\cos^2\theta-\sin^2\theta}{1}\mathrm{d}\theta$

They must have replaced dx with $\mathrm{d}\theta$ the correct way round for this mark, so if $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mathrm{g}(\theta)$ then they must have $\mathrm{d}x = \mathrm{g}(\theta)\{\mathrm{d}\theta\}$ and **not** $\mathrm{d}x = \frac{1}{\mathrm{g}(\theta)}\{\mathrm{d}\theta\}$

It is acceptable to replace e.g. $\tan^2\theta\cos^2\theta$ with $\sin^2\theta$
Dependent on the previous method mark.

A1*: cso Arrives at $\int \cos 2\theta\,\mathrm{d}\theta$ with sufficient working shown and no incorrect work ignoring limits. The final line must be fully correct, including the integral sign and $\mathrm{d}\theta$ (ignoring limits). Note that this may be seen in part (c) and may score the mark.
All trigonometric identities must be fully correct.
Condone one or two slips in a missing θ but not frequent omissions.
Condone missing integral signs in their intermediate work, but it must be present on the final line.
dx must be replaced with $\mathrm{d}\theta$ at some stage before the final line but does not need to be present in every intermediate line of working.
Condone notational errors throughout e.g. $\sin\theta^2$ provided they are recovered.
This mark is independent from any work to do with limits, i.e., B0B1M1dM1A1* is possible.

Question	Scheme	Marks	AOs
15(c)	$\int \cos 2\theta \, \mathrm{d}\theta = \frac{1}{2}\sin 2\theta$	M1	1.1b
	$\left[\frac{1}{2}\sin 2\theta\right]_{"-\frac{\pi}{4}"}^{"\frac{\pi}{4}"} = \frac{1}{2}\sin\left(2\times\frac{\pi}{4}\right) - \frac{1}{2}\sin\left(2\times-\frac{\pi}{4}\right)$	dM1	1.1b
	$=1$	A1	2.1
		(3)	
			(13 marks)

Notes

(c) **Mark parts (b) and (c) together.**

M1: $\int \cos 2\theta \, \mathrm{d}\theta \rightarrow \pm\frac{1}{2}\sin 2\theta \{+c\}$

dM1: Substitutes their changed limits (not 1 and –1) into $\pm\frac{1}{2}\sin 2\theta$ and subtracts either way round.

May be implied e.g. $\frac{1}{2}+\frac{1}{2}$ provided they have the correct limits.

Dependent on the previous method mark.

A1: Area of 1 found following correct work. This mark requires clear substitution of the correct limits into $\frac{1}{2}\sin 2\theta$ the correct way round. If they have the limits the wrong way round and achieve an answer of –1 they cannot just make the answer positive for this mark.

They may use limits from 0 to $\frac{\pi}{4}$ and multiply their result by 2.

$\frac{3\pi}{4}$ may be used in place of $-\frac{\pi}{4}$ as the lower limit which is acceptable.

Condone any spurious integral symbol accompanying $\frac{1}{2}\sin 2\theta$

The substitution may be implied by e.g. $\frac{1}{2}--\frac{1}{2}$ or $\frac{1}{2}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$ but must follow sight of $\frac{1}{2}\sin 2\theta$

Condone use of limits –45 and 45 (degrees) the correct way round for full marks in part (c).
Note that use of a calculator on the original integral will give the correct answer. An answer of 1 scores no marks without evidence of scoring both method marks above including the substitution.

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Question	Scheme	Marks	AOs
10(a)	$8(4)-4^{\frac{5}{2}}=32-32=0$	B1	1.1b
		(1)	
(b)	$8-\frac{5}{2}x^{\frac{3}{2}}$	B1	1.1b
	$x=4\Rightarrow\left\{\frac{dy}{dx}=\right\}8-\frac{5}{2}\times 8=-12$ $\Rightarrow y\{-0\}=\text{"}-12\text{"}(x-4)$	M1	1.1b
	$12x+y=48$ *	A1*	1.1b
		(3)	
(c)	Attempts to find one of the coordinates of the point of intersection $y=8x,\ 12x+y=48\Rightarrow y=19.2$ (or $x=2.4$)	M1	1.1b
	Triangle area is $\frac{1}{2}\times 4\times\text{"19.2"}\left(=38.4 \text{ or } \frac{192}{5}\right)$ or $\int_0^{\text{"2.4"}} 8x\ dx+\int_{\text{"2.4"}}^{4}\text{"}(48-12x)\text{"}\ dx$	dM1	3.1a
	$\int\left(8x-x^{\frac{5}{2}}\right)dx=4x^2-\frac{2}{7}x^{\frac{7}{2}}$	B1	1.1b
	$A=38.4-\left[\text{"}4x^2-\frac{2}{7}x^{\frac{7}{2}}\text{"}\right]_0^4=38.4-64+\frac{256}{7}$	ddM1	3.1a
	$=\frac{384}{35}$	A1	1.1b
		(5)	
			(9 marks)

Notes

(a)

B1: Substitutes $x=4$ into the equation of the curve and verifies that $y=0$. Accept "$8(4)-4^{\frac{5}{2}}=0$"

Alternatively, sets $8x-x^{\frac{5}{2}}=0$ and solves with correct processing to achieve $x=4$.

As a minimum accept e.g. $8x-x^{\frac{5}{2}}=0\ \Rightarrow\ x^{\frac{3}{2}}=8\ \Rightarrow\ \{x=\}\ 4$ which may follow factorisation.

(b)

B1: Correct differentiation. The $\frac{dy}{dx}=$ need not be present.

M1: Correct method for finding the equation of the tangent at $A(4, 0)$.

Requires substitution of $x=4$ into their $\frac{dy}{dx}$ and an attempt at the equation of the line using this gradient. If using $y-y_1=m(x-x_1)$ then condone the omission of the – 0.

If $y=mx+c$ is used they must proceed as far as $c=\ldots$

Accept $\frac{dy}{dx}=-12$ or $m=-12$ without explicit substitution of $x=4$ provided $8-\frac{5}{2}x^{\frac{3}{2}}$ is seen.

A1*: Correct work leading to the given equation having scored B1M1.
Condone $y+12x=48$ and apply isw once seen.
Do not condone $12x+y-48=0$ (unless a correct equation = 48 is seen).

(c) **Note:** Condone poor notation such as missing dx or spurious $\int$ symbols throughout.

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2
You may need to check the diagram or limits to their integrals.

dM1: Correct method for the area of the triangle. e.g. Triangle area is $\frac{1}{2}\times 4\times "19.2"\left(=38.4 \text{ or } \frac{192}{5}\right)$

If integration is attempted then condone slips in their rearrangement of $12x+y=48$ to $y=48-12x$ and note that their integrals do need not to be evaluated, so for example

look for $\int_0^{"2.4"} 8x\ \mathrm{d}x+\int_{"2.4"}^{4} "(48-12x)"\ \mathrm{d}x \qquad \left\{=\frac{576}{25}+\frac{384}{25}=23.04+15.36\right\}$

B1: Correct integration of **curve** ignoring limits, i.e. $4x^2-\frac{2}{7}x^{\frac{7}{2}}$ but condone e.g. $\frac{8x^{1+1}}{2}-\frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.
Does not need to reach a value. Dependent on both previous M marks.
Implied by $38.4-\frac{192}{7}$ or a correct final answer $\frac{384}{35}$
Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0 unless there is evidence of substitution (which need not be evaluated).

A1: Correct exact value. Either $\frac{384}{35}$ or $10\frac{34}{35}$

Alternative using lines – curve:

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2
You may need to check the diagram or limits to their integrals.

dM1: Correct method for at least one part (0 to "2.4" **or** "2.4" to 4) of the area of R including limits.
Condone slips in their rearrangement of $12x+y=48$ to $y=48-12x$ and note that their integrals do need not to be evaluated, so for example

look for $\int_0^{"2.4"} 8x-\left(8x-x^{\frac{5}{2}}\right)\mathrm{d}x$ or $\int_{"2.4"}^{4} "(48-12x)"-\left(8x-x^{\frac{5}{2}}\right)\ \mathrm{d}x$ (or a sum of both)

B1: Correct integration of **both** regions ignoring limits. May be completed as a sum or separately.

Condone e.g. $\frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$ in place of $\frac{2}{7}x^{\frac{7}{2}}$ Note that each integral may have been simplified.

$\int_{\dots}^{\dots} x^{\frac{5}{2}}\ \mathrm{d}x$ and {+} $\int_{\dots}^{\dots} 48-20x+x^{\frac{5}{2}}\ \mathrm{d}x \rightarrow \left[\frac{2}{7}x^{\frac{7}{2}}\right]_{\dots}^{\dots}$ and {+} $\left[48x-10x^2+\frac{2}{7}x^{\frac{7}{2}}\right]_{\dots}^{\dots}$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.
Does not need to reach a value. Dependent on both previous M marks.
This approach requires:

- substitution of 0, their 2.4 and 4 in the correct places
- the $\frac{2}{7}(2.4)^{\frac{7}{2}}-\frac{2}{7}(2.4)^{\frac{7}{2}}$ to be cancelled (may be implied by a correct final answer $\frac{384}{35}$)

Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0 unless there is evidence that the $\frac{2}{7}(2.4)^{\frac{7}{2}}-\frac{2}{7}(2.4)^{\frac{7}{2}}$ has been cancelled e.g. ~~6.118...~~ – ~~6.118...~~

A1: Correct exact value. Either $\frac{384}{35}$ or $10\frac{34}{35}$

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
13(a)	$x = a\sin^2\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2a\sin\theta\cos\theta$	B1	1.1b
	$\int x^{\frac{1}{2}}\sqrt{a-x}\,\mathrm{d}x = \int \sqrt{a}\sin\theta\sqrt{a-a\sin^2\theta}\times 2a\sin\theta\cos\theta\{\mathrm{d}\theta\}$	M1	2.1
	$= \int \sqrt{a}\sin\theta\sqrt{a}\cos\theta\times 2a\sin\theta\cos\theta\,\mathrm{d}\theta = 2a^2\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$ $= 2a^2\int\left(\frac{1}{2}\sin 2\theta\right)^2\mathrm{d}\theta$	dM1	3.1a
	Replaces or considers limits $\{x=0\Rightarrow\}\,\theta=0,\ \{x=a\Rightarrow\}\,\theta=\dfrac{\pi}{2}$ $= \frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta$ *	A1*	1.1b
		(4)	
(b)	$\ldots\int \sin^2 2\theta\,\mathrm{d}\theta \rightarrow \ldots\int \frac{1-\cos 4\theta}{2}\mathrm{d}\theta$	M1	1.1b
	$\rightarrow \ldots\left(\frac{\theta}{2}-\frac{1}{8}\sin 4\theta\right)$	dM1	2.1
	$\mu\int \sin^2 2\theta\,\mathrm{d}\theta \rightarrow \frac{\mu}{2}\left(\theta-\frac{1}{4}\sin 4\theta\right)$	A1	1.1b
	$\left\{\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\,\mathrm{d}x\right\} = \frac{a^2}{4}\left(\frac{\pi}{2}-0-0\right) = \frac{1}{8}\pi a^2$	A1	1.1b
		(4)	
			(8 marks)

Notes

(a)

B1: $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2a\sin\theta\cos\theta$ or $\mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta$ o.e. seen or implied by their substitution.

Note that writing $x = a\sin^2\theta = \frac{a}{2}(1-\cos 2\theta) \rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = a\sin 2\theta$ is correct. Condone use of $\dfrac{\mathrm{d}x}{\mathrm{d}a}$

M1: Attempts to substitute, fully replacing $x^{\frac{1}{2}}$ and $\sqrt{a-x}$ with θ's and dx with their dx = …

Look for $x^{\frac{1}{2}}\sqrt{a-x}\,\mathrm{d}x \rightarrow \mathrm{f}(\theta)\mathrm{g}(\theta)\mathrm{h}(\theta)$ where

- $\mathrm{f}(\theta)$ is an attempt at $\sqrt{a\sin^2\theta}$ e.g. allow $a\sin\theta$ but just $a\sin^2\theta$ is not condoned
- $\mathrm{g}(\theta)$ is an attempt at $\sqrt{a-a\sin^2\theta}$ but not $\sqrt{a}-\sqrt{a\sin^2\theta}$ unless $\sqrt{a-a\sin^2\theta}$ is attempted first
- $\mathrm{h}(\theta)$ = their dx or their $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ or $\dfrac{1}{\text{their }\frac{\mathrm{d}x}{\mathrm{d}\theta}}$ (in terms of θ only but condone da seen)

Condone slips provided the intention is clear, e.g. $x^{\frac{1}{2}} \rightarrow \sqrt{a}\sin^2\theta$ but x must be eliminated. There is no need to replace the limits and condone the absence of dθ and/or the integral sign for this mark.

dM1: Attempts to use $\sin 2\theta = 2\sin\theta\cos\theta$ to convert an integral of the form $\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$ or

e.g. the form $\int \sin\theta\cos\theta\sin 2\theta\,\mathrm{d}\theta$ to $\int ...\sin^2 2\theta\,\mathrm{d}\theta$

There is no need to replace the limits and condone the absence of $\mathrm{d}\theta$ and/or the integral sign for this mark.

A1*: Replaces or considers limits $\{x=0\Rightarrow\}\theta=0,\ \{x=a\Rightarrow\}\theta=\dfrac{\pi}{2}$ at some stage before the given answer and proceeds with no errors to the given answer. The replaced limits may appear with their integral symbol and do not have to be justified and do not have to appear on every line. Condone infrequent slips in notation, e.g. $\sin\theta^2$ in a line as long as it is not consistently poor. You must see the integral sign with the correct limits and the $\mathrm{d}\theta$ together in the given answer.

(b)

M1: Adopts an appropriate strategy by using the double angle identity to obtain an integrable form $...\int \sin^2 2\theta\,\mathrm{d}\theta \to ...\int \dfrac{\pm1\pm\cos 4\theta}{2}\mathrm{d}\theta$ which may be seen as

$\lambda\int \sin^2 2\theta\,\mathrm{d}\theta \to \dfrac{\lambda}{2}\int \pm1\pm\cos 4\theta\,\mathrm{d}\theta$ with the $\dfrac{1}{2}$ absorbed into their coefficient of the integral.

dM1: Integrates into the form $\pm p\theta \pm q\sin 4\theta$

A1: Correct integration of $\mu\int \sin^2 2\theta\,\mathrm{d}\theta$ to $\dfrac{\mu}{2}\left(\theta-\dfrac{1}{4}\sin 4\theta\right)$. Here μ may be 1.

Condone lack of limits here.

A1: Applies limits to the correct integral and proceeds to $\dfrac{1}{8}\pi a^2$ following correct work.

There is no need to see 0 substituted in and condone any omission of integral signs and/or $\mathrm{d}\theta$

Note that $\dfrac{a^2}{4}\left[\theta+\dfrac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}=\dfrac{1}{8}\pi a^2$ is incorrect and scores M1dM1A0A0

Use of $\sin^2 2\theta=\dfrac{\pm1\pm\cos k\theta}{2}$ with $k\neq 4$ scores M0dM0A0A0 but may lead to $\dfrac{1}{8}\pi a^2$

Condone use of x in place of θ e.g. $\dfrac{a^2}{4}\left[x-\dfrac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}=\dfrac{1}{8}\pi a^2$

See overleaf for some alternative approaches.

Alternative 13(a) working backwards:

$\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta$ $\qquad$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=2a\sin\theta\cos\theta$ score B1 (as in main scheme)

$=\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}(2\sin\theta\cos\theta)(2\sin\theta\cos\theta)\,\mathrm{d}\theta$

$=a\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,\mathrm{d}x$

Score M1 here for using the double angle identity and replacing $\ldots\sin\theta\cos\theta\,\mathrm{d}\theta$ with $\mathrm{d}x$

$=a\int_0^{a}\sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\,\mathrm{d}x$

Score dM1 here for a full attempt to replace all trig leading to everything in terms of x only

Must come from the form $\int\ldots\sin\theta\cos\theta\,\mathrm{d}x$

A1 fully correct with limits replaced / considered before the final line and the final line fully correct with limits, integral sign and dx as per the main scheme.

Alternative 13(b) via IBP Way 1:

$\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta$ $\qquad$ $\left\{\begin{array}{ll}u=\sin^2 2\theta & v'=1\\ u'=4\sin 2\theta\cos 2\theta & v=\theta\end{array}\right\}$

$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-4\int_0^{\frac{\pi}{2}}\theta\sin 2\theta\cos 2\theta\,\mathrm{d}\theta$

$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\theta\sin 4\theta\,\mathrm{d}\theta$ $\qquad$ $\left\{\begin{array}{ll}u=\theta & v'=\sin 4\theta\\ u'=1 & v=-\frac{\cos 4\theta}{4}\end{array}\right\}$

$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\left(\left[-\frac{\theta\cos 4\theta}{4}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\frac{\cos 4\theta}{4}\,\mathrm{d}\theta\right)$ $\qquad$ Score M1 here.

$=\left[\theta\sin^2 2\theta\right]_0^{\frac{\pi}{2}}-2\left[-\frac{\theta\cos 4\theta}{4}+\frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{2}}$ Score dM1 here, A1 if correct (ignoring limits).

$=\left[\theta\sin^2 2\theta+\frac{\theta\cos 4\theta}{2}-\frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}}$

$=\left(0+\frac{\pi}{4}-0\right)-(0+0-0)=\frac{\pi}{4}\ \rightarrow\ \left\{\int_0^{a}x^{\frac{1}{2}}\sqrt{a-x}\,\mathrm{d}x=\right\}=\frac{1}{8}\pi a^2$ score A1* (with no errors).

Alternative 13(b) via IBP Way 2:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta \qquad \left\{ \begin{array}{ll} u = \sin 2\theta & v' = \sin 2\theta \\ u' = 2\cos 2\theta & v = -\dfrac{\cos 2\theta}{2} \end{array} \right\}$$

$$= \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^2 2\theta \, \mathrm{d}\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 2\theta \, \mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \, \mathrm{d}\theta$$ Score M1 here.

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} + \theta \right]_0^{\frac{\pi}{2}}$$ Score dM1 here, A1 if correct including the 2 (ignoring limits).

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, \mathrm{d}\theta = \frac{1}{2}\left(0 + \frac{\pi}{2} \right) - \frac{1}{2}(0+0) = \frac{\pi}{4}$$

$$\rightarrow \left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, \mathrm{d}x = \right\} = \frac{1}{8}\pi a^2$$ score A1* (with no errors).

Question	Scheme	Marks	AOs

11	$\int 8x^2 e^{-3x}\,dx = -\frac{8x^2}{3}e^{-3x} + \int \frac{16x}{3}e^{-3x}\,dx$	M1 A1	2.1 1.1b
	$= -\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} + \int \frac{16}{9}e^{-3x}\,dx$	**d**M1	1.1b
	$\left[-\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} - \frac{16}{27}e^{-3x}\right]_0^1$ $= -\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-0-0-\frac{16}{27}\right)$	M1	2.1
	$= \frac{16}{27} - \frac{136}{27}e^{-3}$	A1	1.1b
		(5)	
			(5 marks)

Notes

Mark positively in this question and do not penalise poor notation such as a missing "d*x*" or spurious integral signs, "+ *c*" etc. as long as the intention is clear.

M1: Obtains $\pm\alpha x^2 e^{-3x} \pm \beta\int x\,e^{-3x}\,dx$

(you do not need to be concerned about how they arrive at this)

A1: Correct expression **simplified or unsimplified**. E.g. allow $-\frac{8x^2}{3}e^{-3x} - \int -\frac{16x}{3}e^{-3x}\,dx$

Note that we condone the "8" missing for this mark so allow e.g. $-\frac{x^2}{3}e^{-3x} - \int -\frac{2x}{3}e^{-3x}\,dx$

Note that notation may be poor here but the intention clear e.g. if they obtain $-\frac{8x^2}{3}e^{-3x} + \left[\frac{16x}{3}e^{-3x}\right]$ and then attempt to integrate $\frac{16x}{3}e^{-3x}$ both marks can be implied.

dM1: Attempts parts again on $\pm\beta\int x\,e^{-3x}\,dx$ to obtain $\pm Axe^{-3x} \pm B\int e^{-3x}\,dx$

This may be seen in isolation and does not need to be seen as part of the complete integration. **Depends on the first method mark.**

Watch for the DI method (with or without the 8):

	D	**I**
+	$8x^2$	e^{-3x}
–	$16x$	$-\frac{1}{3}e^{-3x}$
+	16	$\frac{1}{9}e^{-3x}$
–	0	$-\frac{1}{27}e^{-3x}$

Giving the correct integration e.g. $\int 8x^2 e^{-3x}\,dx = -\frac{8x^2}{3}e^{-3x} - \frac{16x}{9}e^{-3x} - \frac{16}{27}e^{-3x}$

In such cases score M1dM1 for obtaining $\pm px^2e^{-3x} \pm qxe^{-3x} \pm re^{-3x}$, $p,q,r \neq 0$ and then A1 for the **correct** first 2 terms, with or without the factor of 8.

Note that for this approach M1A1dM0 is not possible.

M1: Substitutes the limits 1 and 0 into an expression of the form $\pm\alpha x^2 e^{-3x} \pm \beta x e^{-3x} \pm \gamma e^{-3x},\ \alpha, \beta, \gamma \neq 0$ and subtracts the right way round.
Must see evidence of the use of **both** limits and subtraction and use of $e^0 = 1$.
Note that some candidates apply the limits as they go e.g. to the $\left[-\frac{8x^2}{3}e^{-3x}\right]$ which is acceptable but you will need to check carefully that overall they are satisfying the conditions above.
Condone not realising that the first 2 terms evaluate to 0 when substituting $x = 0$ e.g. condone $-\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-\frac{8}{3} - \frac{16}{9} - \frac{16}{27}\right)$ as we have evidence of $e^0 = 1$
Note that e.g. $-\frac{8}{3}e^{-3} - \frac{16}{9}e^{-3} - \frac{16}{27}e^{-3} - \left(-\frac{16}{27}e^{0}\right) = -\frac{136}{27}e^{-3} - \frac{16}{27}$ scores M0 as it suggests that $e^0 = -1$ not +1.

A1: Correct answer of $\frac{16}{27} - \frac{136}{27}e^{-3}$ but allow equivalent exact fractions and condone $\frac{16}{27} - \frac{136}{27e^3}$. Isw once the correct answer is seen.

Candidates who consistently misread $8x^2e^{-3x}$ as $8x^2e^{3x}$:

$$\int 8x^2e^{3x}\,dx = \frac{8x^2}{3}e^{3x} - \int \frac{16x}{3}e^{3x}\,dx$$
$$= \frac{8x^2}{3}e^{3x} - \frac{16x}{9}e^{3x} + \int \frac{16}{9}e^{3x}\,dx$$
$$\left[\frac{8x^2}{3}e^{3x} - \frac{16x}{9}e^{3x} + \frac{16}{27}e^{3x}\right]_0^1$$
$$= \frac{8}{3}e^3 - \frac{16}{9}e^3 + \frac{16}{27}e^3 - \left(\frac{16}{27}\right) = \frac{40}{27}e^3 - \frac{16}{27}$$

Scores a maximum of M1A0dM1M1A0
The main scheme can be applied similarly e.g.

M1: Attempts parts to obtain $\alpha x^2e^{3x} - \beta\int x\,e^{3x}\,dx,\quad \alpha, \beta > 0$

A0: Not available

dM1: Attempts parts again on $\beta\int x\,e^{3x}\,dx$ to obtain $Cxe^{3x} - D\int e^{3x}\,dx,\ C, D > 0$

M1: Substitutes the limits 1 and 0 into an expression of the form $\pm\lambda x^2e^{3x} \pm \mu x e^{3x} \pm \gamma e^{3x},\ \lambda, \mu, \gamma \neq 0$ and subtracts the right way round.
Must see evidence of the use of **both** limits and subtraction and use of $e^0 = 1$.

A0: Not available

But note, do **not** allow mixing of $3x$'s and $-3x$'s. If there are a mixture, apply the main scheme.

Question	Scheme	Marks	AOs

1	$x^{\frac{1}{2}}(2x-5)=...x^{\frac{3}{2}}+...x^{\frac{1}{2}}$ or $\frac{x^{\frac{1}{2}}(2x-5)}{3}=\frac{...x^{\frac{3}{2}}+...x^{\frac{1}{2}}}{3}$	M1	1.1b
	$\frac{2x^{\frac{3}{2}}}{3}-\frac{5x^{\frac{1}{2}}}{3}$	A1	1.1b
	$\int \frac{2x^{\frac{3}{2}}}{3}-\frac{5x^{\frac{1}{2}}}{3}\,\mathrm{d}x=...x^{\frac{5}{2}}\pm...x^{\frac{3}{2}}\ (+c)$	dM1	1.1b
	$\frac{4}{15}x^{\frac{5}{2}}-\frac{10}{9}x^{\frac{3}{2}}+c$	A1	1.1b
		(4)	
			(4 marks)

Notes

M1: Attempts to multiply out the brackets of the numerator and either writes the expression (or just the numerator) as a sum of terms with **indices**. Award for either one correct index of $...x^{\frac{3}{2}}+...x^{\frac{1}{2}}$ which comes from a correct method. Condone appearing as terms on separate lines for this mark. The correct index may be implied later when e.g. $\sqrt{x}\rightarrow x^{\frac{3}{2}}$ or $\left(\sqrt{x}\right)^3\rightarrow x^{\frac{5}{2}}$ after they have integrated.

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: $\frac{2x^{\frac{3}{2}}}{3}-\frac{5x^{\frac{1}{2}}}{3}$ or equivalent e.g. $\frac{1}{3}\left(2x^{\frac{3}{2}}-5x^{\frac{1}{2}}\right)$. Condone $\frac{2x^{\frac{3}{2}}-5x^{\frac{1}{2}}}{3}$ May be implied by further work.

The correct index may be implied later when e.g. $\sqrt{x}\rightarrow x^{\frac{3}{2}}$ or $\left(\sqrt{x}\right)^3\rightarrow x^{\frac{5}{2}}$ after they have integrated.

Ignore incorrect integration notation around the terms. Ignore any presence or absence of dx.

Be aware that a factor of $\frac{1}{3}$ may be taken outside of the integral so you may need to look at further work to award the first A mark if work on the two terms is done separately or in a list.

May be unsimplified and the two terms may appear in a list which is fine.

Coefficients must be exact.

dM1: Increases the power by one on an x^n term where n is a fraction. The index does not need to be processed.

e.g. $...x^{\frac{3}{2}+1}$ or $...x^{\frac{1}{2}+1}$ It is dependent on the previous method mark so at least one of the terms must have had a correct index.

Note that integrating the numerator and denominator e.g. $\frac{2x^{\frac{3}{2}}}{3}-\frac{5x^{\frac{1}{2}}}{3}\rightarrow\frac{...x^{\frac{5}{2}}}{3x}-\frac{...x^{\frac{3}{2}}}{3x}$ is dM0.

A1: $\frac{4}{15}x^{\frac{5}{2}}-\frac{10}{9}x^{\frac{3}{2}}+c$ and including the constant or simplified exact equivalent such as

$\frac{4}{15}\sqrt{x^5}-\frac{10}{9}\sqrt{x^3}+c$ or $\frac{4}{15}x^{2.5}-\frac{10}{9}x^{1.5}+c$ or $\frac{1}{45}\left(12x^{\frac{5}{2}}-50x^{\frac{3}{2}}\right)+c$ or $\frac{x^{\frac{3}{2}}}{45}(12x-50)+c$. Fractions must be in their lowest terms and indices processed.

Do not accept e.g. $0.267x^{\frac{5}{2}}-1.11x^{\frac{3}{2}}+c$ but allow $0.2\dot{6}x^{\frac{5}{2}}-1.\dot{1}x^{\frac{3}{2}}+c$

Isw once a correct answer is seen but withhold this mark if there is spurious notation around their final answer e.g. $\int\frac{4}{15}x^{\frac{5}{2}}-\frac{10}{9}x^{\frac{3}{2}}+c\ \mathrm{d}x$ is M1A1dM1A0

Alternative method using integration by parts example

M1: e.g. $\int x^{\frac{1}{2}}(2x-5)\,\mathrm{d}x=...x^{\frac{3}{2}}(2x-5)-\int...x^{\frac{3}{2}}\,\mathrm{d}x$ (applies integration by parts correctly to typically achieve this form – the $(2x-5)$ may also be split up as well – send to review if unsure how to mark)

This may also be done the other way round e.g. $\int x^{\frac{1}{2}}(2x-5)\,\mathrm{d}x=...x^{\frac{1}{2}}(x^2-5x)-\int...x^{\frac{3}{2}}\pm...x^{\frac{1}{2}}\,\mathrm{d}x$

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: A correct intermediate stage applying integration by parts with correct coefficients.

e.g. $\int\frac{x^{\frac{1}{2}}(2x-5)}{3}\,\mathrm{d}x=\frac{2}{3}x^{\frac{3}{2}}\left(\frac{2x-5}{3}\right)-\int\frac{4}{9}x^{\frac{3}{2}}\,\mathrm{d}x$ (or unsimplified equivalent).

Coefficients must be exact. (See main scheme notes above) The other way round this could appear as

e.g. $\int\frac{x^{\frac{1}{2}}(2x-5)}{3}\,\mathrm{d}x=\frac{1}{3}x^{\frac{1}{2}}(x^2-5x)-\frac{1}{6}\int x^{\frac{3}{2}}-5x^{\frac{1}{2}}\,\mathrm{d}x$. Condone a missing d$x$. May be implied.

dM1: Increases the power by one on an x^n term where n is a fraction e.g. $\int...x^{\frac{3}{2}}\,\mathrm{d}x\rightarrow...x^{\frac{5}{2}}$ The index does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}}-\frac{10}{9}x^{\frac{3}{2}}+c$ or exact simplified equivalent. (See main scheme notes above)

Alternative method using the substitution method

M1: e.g. let $u=x^{\frac{1}{2}}\Rightarrow\int...u^4+...u^2\,\mathrm{d}u$ (uses a substitution to express the integral in terms of another variable. Allow slips with the coefficients, but the indices should be correct for their substitution)

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: e.g. $\int\frac{4u^4}{3}-\frac{10u^2}{3}\,\mathrm{d}u$ or unsimplified equivalent. Coefficients must be exact. See main scheme notes above). May be implied by further work. Condone a missing dx.

dM1: $\int...u^4+...u^2\,\mathrm{d}u\rightarrow...u^5+...u^3$ (increases the power by one on at least one of their indices – does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}}-\frac{10}{9}x^{\frac{3}{2}}+c$ or exact simplified equivalent. (See main scheme notes above)

There may be alternative substitutions, but the same marking principles apply.

Question	Scheme	Marks	AOs

10(a)	e.g. $\dfrac{3kx-18}{(x+4)(x-2)} \equiv \dfrac{A}{x+4}+\dfrac{B}{x-2} \Rightarrow 3kx-18 \equiv A(x-2)+B(x+4)$ **or** $\dfrac{3kx-18}{(x+4)(x-2)} \equiv \dfrac{A}{x-2}+\dfrac{B}{x+4} \Rightarrow 3kx-18 \equiv A(x+4)+B(x-2)$	**B1**	1.1b
	$6k-18=6B \Rightarrow B=\ldots$ or $-12k-18=-6A \Rightarrow A=\ldots$ **or** $3kx-18 \equiv (A+B)x+4B-2A \Rightarrow A+B=3k,\ -18=4B-2A$ $\Rightarrow A=\ldots$ or $B=\ldots$	**M1**	1.1b
	$\dfrac{2k+3}{x+4}+\dfrac{k-3}{x-2}$	**A1**	1.1b
		(3)	
(b)	$\int \left(\dfrac{"2k+3"}{x+4}+\dfrac{"k-3"}{x-2}\right) \mathrm{d}x = \ldots\ln(x+4)+\ldots\ln(x-2)$	**M1**	1.2
	$("2k+3")\ln(x+4)+("k-3")\ln(x-2)$	**A1ft**	1.1b
	$("2k+3")\ln(5)-("k-3")\ln(5) \Rightarrow ("k+6")\ln 5=21 \Rightarrow k=\ldots$	**dM1**	3.1a
	$(k=)\dfrac{21}{\ln 5}-6$	**A1**	2.2a
		(4)	
			(7 marks)

Notes

(a)

B1: Correct form for the partial fractions and sets up the correct corresponding identity which may be implied by two equations in A and B if they are comparing coefficients.

M1: Either

- substitutes $x=2$ or $x=-4$ in an attempt to find A or B in terms of k
- expands the rhs, collects terms and compares coefficients in an attempt to find A or B in terms of k

Or may be implied by one correct fraction (numerator **and** denominator)

You may see candidates substituting two other values of x and then solving simultaneous equations.

A1: Achieves $\dfrac{2k+3}{x+4}+\dfrac{k-3}{x-2}$ with no errors. Must be the correct partial fractions not just for correct numerators. May be seen in (b). Correct answer implies B1M1A1. One correct fraction only B0M1A0

(b)

M1: Attempts to find $\int\left(\frac{"2k+3"}{x+4}+\frac{"k-3"}{x-2}\right)\mathrm{d}x$. Score for either $\frac{\ldots}{x+4}\to\ldots\ln(x+4)$ or $\frac{\ldots}{x-2}\to\ldots\ln(x-2)$

Allow the … to be in terms of k or just constants but there must be no x terms.

Condone invisible brackets for this mark.

A1ft: $("2k+3")\ln|x+4|+("k-3")\ln|x-2|$

but condone round brackets e.g. $("2k+3")\ln(x+4)+("k-3")\ln(x-2)$ or equivalent e.g. $("2k+3")\ln(x+4)+("k-3")\ln(2-x)$

Follow through their partial fractions with numerators which must both be in terms of k.

Condone missing brackets as long as they are recovered later e.g. when applying limits.

dM1: A full attempt to find the value of k. To score this mark they must have attempted to integrate their partial fractions, substituted in the correct limits, subtracted either way round, set = 21 and attempted to solve to find k. Condone omission of the terms containing ln(1) or ln(-1).

Note that e.g. ln(-5) or ln(5) must be seen but may be disregarded **after** substitution and subtraction.

Do not be concerned with the processing as long as they proceed to $k=\ldots$

Condone if they use x instead of k after limits have been used as long as the intention is clear.

A1: Deduces $(k=)\frac{21}{\ln 5}-6$ or exact equivalent e.g. $\frac{21-6\ln 5}{\ln 5}$, $\frac{21-3\ln 25}{\ln 5}$.

Allow recovery from expressions that contain e.g. ln(−5) as long as it is dealt with subsequently.

Also allow recovery from invisible brackets. Condone $x=\frac{21}{\ln 5}-6$

Some candidates may use substitution in part (b) e.g.

$$\int\left(\frac{"2k+3"}{x+4}+\frac{"k-3"}{x-2}\right)\mathrm{d}x=\int\left(\frac{"2k+3"}{x+4}\right)\mathrm{d}x+\int\left(\frac{"k-3"}{x-2}\right)\mathrm{d}x$$

$$u=x+4\Rightarrow\int\left(\frac{"2k+3"}{x+4}\right)\mathrm{d}x=\int\left(\frac{"2k+3"}{u}\right)\mathrm{d}u=\ldots\ln u$$

$$u=x-2\Rightarrow\int\left(\frac{"k-3"}{x-2}\right)\mathrm{d}x=\int\left(\frac{"k-3"}{u}\right)\mathrm{d}u=\ldots\ln u$$

Score **M1** for integrating at least once to an appropriate form as in the main scheme e.g. …ln$u$

A1ft: For $("2k+3")\ln|u|+("k-3")\ln|u|$

but condone $("2k+3")\ln u+("k-3")\ln u$ which may be seen separately

Follow through their "A" and "B" in terms of k.

Condone missing brackets as long as they are recovered later e.g. when applying limits.

dM1: A full attempt to find the value of k. To score this mark they must have attempted to integrate their partial fractions using substitution, substituted in the correct changed limits and subtracts either way

round, set = 21 and attempted to solve to find k. Do not be concerned with processing as long as they proceed to $k = \ldots$ Condone omission of terms which contain e.g. ln(1) or ln(-1).

Note that e.g. ln(-5) or ln(5) must be seen but may be disregarded **after** substitution and subtraction.

$\left[(2k+3)\ln u\right]_{1}^{5} + \left[(k-3)\ln u\right]_{-5}^{-1} = 21 \Rightarrow (2k+3)\ln 5 - (2k+3)\ln 1 + (k-3)\ln 1 - (k-3)\ln 5 = 21$

$\Rightarrow (2k+3)\ln 5 - (k-3)\ln 5 = 21 \Rightarrow \left(k+6\right)\ln 5 = 21 \Rightarrow k = \ldots$

A1: $k = \dfrac{21}{\ln 5} - 6$ or exact equivalent e.g. $\dfrac{21 - 6\ln 5}{\ln 5}$, $\dfrac{21 - 3\ln 25}{\ln 5}$, $21\log_5 \mathrm{e} - 6$.

Allow recovery from expressions that contain e.g. ln(–5) as long as it is dealt with subsequently.

Also allow recovery from invisible brackets.

Question	Scheme	Marks	AOs

13(a)	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=3^{-2}(1+...x+...x^2)$	**M1**	1.1b
	$(-2)\left(\frac{x}{3}\right)$ or $\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	**M1**	1.1b
	$\left(1+\frac{x}{3}\right)^{-2}=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	**A1**	1.1b
	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$	**A1**	2.1
		(4)	

(a)

M1: Attempts a binomial expansion by taking out a factor of 3^{-2} or $\frac{1}{3^2}$ or $\frac{1}{9}$ and achieves at least the first 3 terms in their expansion. May be seen separately e.g. $\frac{1}{9}$ and $(1+...x+...x^2)$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)(kx)$ or $\frac{(-2)(-2-1)}{2!}(kx)^2$ where $k\neq 1$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $\left(1+\frac{x}{3}\right)^{-2}$ e.g. $=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2-...$ or $1-\frac{2x}{3}+\frac{x^2}{3}-...$ Do not condone missing brackets unless they are implied by subsequent work.

Condone $\left(-\frac{x}{3}\right)^2$ for $\left(\frac{x}{3}\right)^2$

Also allow this mark for 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

<u>Direct expansion, if seen, should be marked as follows:</u>

$$\left((3+x)^{-2}=3^{-2}-2\times 3^{-3}\times x+\frac{-2(-2-1)}{2!}\times 3^{-4}\times x^2\right)$$

M1: For $(3+x)^{-2}=3^{-2}+3^{-3}\times\alpha x+3^{-4}\times\beta x^2$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)\times 3^{-3}x$ or $\frac{(-2)(-2-1)}{2!}\times 3^{-4}x^2$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $(3+x)^{-2}$ e.g. $3^{-2}-2\times 3^{-3}\times x+\frac{-2(-2-1)}{2!}\times 3^{-4}\times x^2$

Also award for at least 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Note that M0M1A1A0 is a possible mark trait in either method

Note regarding a possible misread in parts (b) and (c)

Some candidates are misreading $\int \frac{6x}{(3+x)^2}\,\mathrm{d}x$ in parts (b) and (c) as $\int \frac{6}{(3+x)^2}\,\mathrm{d}x$

If parts (b) and (c) are consistently attempted with $\int \frac{6}{(3+x)^2}\,\mathrm{d}x$ then we will allow the M marks in (b) **only**. **M1** for $x^n \to x^{n+1}$ applied to their expansion in part (a) or 6× (their expansion in part (a)) and **dM1** for substituting in 0.4 and 0.2 and subtracting either way round (may be implied).

No marks are available in part (c)

MARK PARTS (b) and (c) TOGETHER

(b)	$\int 6x \times "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"\,\mathrm{d}x = \int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = ...$	**M1**	1.1b
	$\int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = \frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}$ oe	**A1**	1.1b
	$\left[" \frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18} "\right]_{0.2}^{0.4} = \left(\frac{(0.4)^2}{3}-\frac{4(0.4)^3}{27}+\frac{(0.4)^4}{18}\right)-\left(\frac{(0.2)^2}{3}-\frac{4(0.2)^3}{27}+\frac{(0.2)^4}{18}\right)$	**dM1**	3.1a
	$= \text{awrt } 0.03304 \text{ or } \frac{223}{6750}$	**A1**	1.1b
		(4)	

(b)

M1: Attempts to multiply their expansion from part (a) by 6*x* or just *x* and attempts to integrate. Condone copying slips and slips in expanding. Look for $x^n \to x^{n+1}$ at least once having multiplied by 6*x* or *x*. Ignore e.g. spurious integral signs.

A1: Correct integration, simplified or unsimplified.

$$\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18} \text{ oe e.g. } \frac{1}{9}\left(3x^2-\frac{4x^3}{3}+\frac{x^4}{2}\right),\ 6\left(\frac{x^2}{18}-\frac{2x^3}{81}+\frac{x^4}{108}\right)$$

If they have extra terms they can be ignored.

Ignore e.g. spurious integral signs.

dM1: An overall problem-solving mark for

- using part (a) by integrating $6x \times$ their binomial expansion and
- substituting in 0.4 and 0.2 and subtracting either way round (may be implied)

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the limits 0.4 and 0.2.

This could be e.g. $\left[\mathrm{f}(x)\right]_{0.2}^{0.4} = ...$ provided the first M was scored. If the integration was correct, evidence can be taken from answer of awrt 0.0330 if limits are not seen elsewhere. **Depends on the first M mark.**

A1: awrt 0.03304 (NB allow the exact value which is $\frac{223}{6750} = 0.033037037...$).

Isw following a correct answer.

Note answers which use additional terms in the expansion to give a different approximation score A0

Also note that the actual value is 0.032865…

Some may use integration by parts in (b) and the following scheme should be applied.

Integration by parts in (b):

Either by taking $u = 6x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"$

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"\mathrm{d}x = 6x \times \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - 6\int\left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right)\mathrm{d}x$$

$$= 6x\left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - \left(\frac{1}{3}x^2 - \frac{2x^3}{27} + \frac{6x^4}{324}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"\mathrm{d}x = kx \times \mathrm{f}(x) - k\int \mathrm{f}(x)\mathrm{d}x = kx \times \mathrm{f}(x) - k\mathrm{g}(x)$$

Where f(x) is an attempt to integrate their expansion from (a) with $x^n \to x^{n+1}$ at least once **and** g(x) is an attempt to integrate their f(x) with $x^n \to x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

Or by taking $u = "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 6x$

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"\mathrm{d}x = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int 3x^2 \times \left(-\frac{2}{27} + \frac{2x}{27}\right)\mathrm{d}x$$

$$= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int\left(\frac{6x^3}{27} - \frac{6x^2}{27}\right)\mathrm{d}x = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \left(\frac{6x^4}{108} - \frac{6x^3}{81}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"\mathrm{d}x = kx^2 \times \mathrm{f}(x) - k\int x^2\mathrm{g}(x)\mathrm{d}x = kx^2 \times \mathrm{f}(x) - k\mathrm{h}(x)$$

Where f(x) is their expansion from (a) and g(x) is an attempt to differentiate their f(x) with $x^n \to x^{n-1}$ at least once **and** h(x) is an attempt to integrate their $x^2\mathrm{g}(x)$ with $x^n \to x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

(c)	Overall problem-solving mark (see notes)	**M1**	3.1a
	$u=3+x \Rightarrow \int_{3.2}^{3.4} \mathrm{f}(u)\,\mathrm{d}u \Rightarrow \int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2}\,\mathrm{d}u \Rightarrow \ldots \ln u + \ldots u^{-1}$	**M1**	1.1b
	$\int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2}\,\mathrm{d}u \Rightarrow 6\ln u + 18u^{-1}$	**A1**	1.1b
	$\left[6\ln u + 18u^{-1}\right]_{3.2}^{3.4} = \left(6\ln 3.4 + \frac{18}{3.4}\right) - \left(6\ln 3.2 + \frac{18}{3.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1
		(5)	

(c) Alt 1	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \frac{\ldots x}{3+x} \pm \ldots \int (3+x)^{-1}\mathrm{d}x = \frac{\ldots x}{3+x} \pm \ldots \ln(3+x)$ oe	**M1**	1.1b
	$= 6\ln(3+x) - \frac{6x}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) - \frac{6(0.4)}{3+0.4}\right) - \left(6\ln(3+0.2) - \frac{6(0.2)}{3+0.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1

(c) Alt 2	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \int \left(\frac{\ldots}{(3+x)} + \frac{\ldots}{(3+x)^2}\right)\mathrm{d}x = \ldots \ln(3+x) + \frac{\ldots}{3+x}$ oe	**M1**	1.1b
	$= 6\ln(3+x) + \frac{18}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) + \frac{18}{3+0.4}\right) - \left(6\ln(3+0.2) + \frac{18}{3+0.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1

(13 marks)

Notes

(c) There are various methods which can be used

M1: An overall problem-solving mark for **all of**

- using an appropriate integration technique e.g. substitution, by parts or partial fractions – note that this may not be correct but mark positively if they have tried one of these approaches
- integrates one of their terms to a natural logarithm, e.g., $\frac{a}{3+x} \to b\ln(3+x)$ or $\frac{a}{u} \to b\ln u$
- substitutes in correct limits and subtracts either way round

M1: Integrates to achieve an expression of the required form for their chosen method

- substitution: $u = x+3 \to \pm\frac{a}{u} \pm b\ln u$ or e.g. $u = (x+3)^2 \to \pm\frac{a}{\sqrt{u}} \pm b\ln u$
- parts: $\pm a\ln(3+x) \pm \frac{bx}{3+x}$ condone missing brackets e.g. $\ldots\ln x + 3$ for $\ldots\ln(3+x)$
- partial fractions: $\pm a\ln(3+x) \pm \frac{b}{3+x}$ condone missing brackets e.g. $\ldots\ln 3 + x$ for $\ldots\ln(3+x)$

A1: Correct integration for their method e.g.

- substitution: $u = x+3 \rightarrow 6\ln u + 18u^{-1}$ or e.g. $u = (x+3)^2 \rightarrow 3\ln u + \dfrac{18}{\sqrt{u}}$
- parts: $6\ln(3+x) - \dfrac{6x}{3+x}$
- partial fractions: $6\ln(3+x) + \dfrac{18}{3+x}$ oe e.g. $3\ln(9+6x+x^2) + \dfrac{18}{3+x}$

Note that the above terms may appear "separated" but must be correct with the correct signs.
(ignore any reference to a constant of integration)
Do not condone missing brackets e.g. $6\ln x + 3$ for $6\ln(3+x)$ unless they are implied by later work.

ddM1: Substitutes in the correct limits for their integral and subtracts either way round to find a value
Depends on both previous method marks.
For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the appropriate limits.
This could be e.g. $\left[\mathrm{f}(x)\right]_{0.2}^{0.4} = ...$ provided both previous M marks were scored.
Note that for substitution they may revert back to 3 + x and so should be using 0.4 and 0.2

A1: A full and rigorous argument leading to $6\ln\left(\dfrac{17}{16}\right) - \dfrac{45}{136}$ or exact equivalent e.g. $3\ln\left(\dfrac{289}{256}\right) - \dfrac{45}{136}$ or e.g. $-6\ln\left(\dfrac{16}{17}\right) - \dfrac{45}{136}$
The brackets are not required around the $\dfrac{17}{16}$ and allow exact equivalents e.g. allow 1.0625 or $1\dfrac{1}{16}$ but not e.g. $\dfrac{3.4}{3.2}$. The $\dfrac{45}{136}$ must be exact or an exact equivalent. Also allow e.g. $6\ln\left|\dfrac{17}{16}\right| - \dfrac{45}{136}$
Ignore spurious integral signs that may appear as part of their solution.

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
16 (a)	Attempts $y.\dfrac{\mathrm{d}x}{\mathrm{d}t}=\left(2\sin 2t+3\sin t\right)\times 16\sin t\cos t$ and uses $\sin 2t=2\sin t\cos t$	M1	2.1
	Correct expanded integrand. **Usually** for one of $\left(R\right)=\int \underline{\underline{48\sin^2 t\cos t+16\sin^2 2t}}\ \mathrm{d}t$ $\left(R\right)=\int \underline{\underline{48\sin^2 t\cos t+64\sin^2 t\cos^2 t}}\ \mathrm{d}t$ $\left(R\right)=\int \underline{\underline{24\sin 2t\sin t+16\sin^2 2t}}\ \mathrm{d}t$	A1	1.1b
	Attempts to use $\cos 4t=1-2\sin^2 2t=\left(1-8\sin^2 t\cos^2 t\right)$	M1	1.1b
	$R=\int_0^a 8-8\cos 4t+48\sin^2 t\cos t\ \mathrm{d}t$ *	A1*	2.1
	Deduces $a=\dfrac{\pi}{4}$	B1	2.2a
		(5)	
(b)	$\int 8-8\cos 4t+48\sin^2 t\cos t\ \mathrm{d}t=8t-2\sin 4t+16\sin^3 t$	M1 A1	2.1 1.1b
	$\left[8t-2\sin 4t+16\sin^3 t\right]_0^{\frac{\pi}{4}}=2\pi+4\sqrt{2}$	M1 A1	2.1 1.1b
		(4)	
			(9 marks)
Notes:			

(a) Condone work in another variable, say $\theta\leftrightarrow t$ if used consistently for the first 3 marks

M1: For the key step in attempting $y.\dfrac{\mathrm{d}x}{\mathrm{d}t}=\left(2\sin 2t+3\sin t\right)\times 16\sin t\cos t$ with an attempt to use $\sin 2t=2\sin t\cos t$ Condone slips in finding $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ but it must be of the form $k\sin t\cos t$

E.g. I $\quad y.\dfrac{\mathrm{d}x}{\mathrm{d}t}=\left(2\sin 2t+3\sin t\right)\times k\sin t\cos t=\left(4\sin t\cos t+3\sin t\right)\times k\sin t\cos t$

E.g. II $\quad y.\dfrac{\mathrm{d}x}{\mathrm{d}t}=\left(2\sin 2t+3\sin t\right)\times k\sin t\cos t=\left(2\sin 2t+3\sin t\right)\times \dfrac{k}{2}\sin 2t$

A1: A correct (expanded) integrand in t . Don't be concerned by the absence of $\int$ or dt or limits

$\left(R\right)=\int \underline{\underline{48\sin^2 t\cos t+16\sin^2 2t}}\ \mathrm{d}t$ or $\left(R\right)=\int \underline{\underline{48\sin^2 t\cos t+64\sin^2 t\cos^2 t}}\ \mathrm{d}t$

but watch for other correct versions such as $\left(R\right)=\int \underline{\underline{24\sin 2t\sin t+16\sin^2 2t}}\ \mathrm{d}t$

If they have the form $P\sin^2 2t$ it is acceptable to write $P\sin^2 2t = \frac{P}{2}\left(\pm 1 \pm \cos 4t\right)$

If they have the form $Q\sin^2 t\cos^2 t$ sight and use of $\sin 2t$ and/or $\cos 2t$ will usually be seen first.

There are many ways to do this, below is such an example

$$Q\sin^2 t\cos^2 t = Q\left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right) = Q\left(\frac{1-\cos^2 2t}{4}\right) = Q\left(\frac{1}{4}-\frac{\cos^2 2t}{4}\right) = Q\left(\frac{1}{4}-\frac{1+\cos 4t}{8}\right)$$

Allow candidates to start with the given answer and work backwards using the same rules. So expect to see $\cos 4t = \pm 1 \pm 2\times \sin^2 2t$ or $\cos 4t = \pm 2\times \cos^2 2t \pm 1$ before double angle identities for $\sin 2t$ or $\cos 2t$ are used.

A1*: Proceeds to the given answer with correct working. The order of the terms is not important. Ignore limits for this mark. The integration sign and the dt must be seen on their final answer. If they have worked backwards there must be a concluding statement to the effect that they know that they have shown it. The integration sign and the dt must also be seen

E.g. Reaches $\int 48\sin^2 t\cos t + 64\sin^2 t\cos^2 t \ \mathrm{d}t$

Answer is $\int 8 - 8\cos 4t + 48\sin^2 t\cos t \ \mathrm{d}t$

$= \int 8 - 8\left(1-2\sin^2 2t\right) + 48\sin^2 t\cos t \ \mathrm{d}t$

$= \int 16\sin^2 2t + 48\sin^2 t\cos t \ \mathrm{d}t$

$= \int 64\sin^2 t\cos^2 t + 48\sin^2 t\cos t \ \mathrm{d}t$ which is the same, ✓

B1: Deduces $a = \frac{\pi}{4}$. It may be awarded from the upper limit and can be awarded from (b)

(b)

M1: For the key process in using a correct approach to integrating the trigonometric terms. May be done separately.

There may be lots of intermediate steps (e.g. let $u = \sin t$) .

There are other more complicated methods so look carefully at what they are doing.

$\int 8 - 8\cos 4t + 48\sin^2 t\cos t \ \mathrm{d}t = ... \pm P\sin 4t \pm Q\sin^3 t$ where P and Q are constants

A1: $\int 8 - 8\cos 4t + 48\sin^2 t\cos t \ \mathrm{d}t = 8t - 2\sin 4t + 16\sin^3 t\left(+c\right)$

If they have written $16\sin^3 t$ as $16\sin t^3$ only award if further work implies a correct answer.

Similarly, 8t may be written as 8x. Award if further work implies 8t, e.g. substituting in their limits. Do not penalise this sort of slip at all, these are intermediate answers.

M1: Uses the limits their a and 0 where $a = \frac{\pi}{6}, \frac{\pi}{4}$ or $\frac{\pi}{3}$ in an expression of the form $kt \pm P\sin 4t \pm Q\sin^3 t$ leading to an exact answer. Ignore evidence at lower limit as terms are 0

A1: CSO $2\pi + 4\sqrt{2}$ or exact **simplified** equivalent such as $2\pi + \frac{8}{\sqrt{2}}$ or $2\pi + \sqrt{32}$.

Be aware that $\int_0^{\frac{\pi}{4}} 8 - 8\cos 4t + 48\sin^2 t\cos t \ \mathrm{d}t = 8t + \lambda\sin 4t + 16\sin^3 t\left(+c\right)$ would lead to the correct answer but would only score M1 A0 M1 A0

5(a)	States or uses $h = 1.5$	B1	1.1a
	Full attempt at the trapezium rule $= \frac{\cdots}{2}\{1.63 + 2.63 + 2\times(2 + 2.26 + 2.46)\}$	M1	1.1b
	= awrt 13.3 or $\frac{531}{40}$	A1	1.1b
		(3)	
(b)(i)	$\int_3^9 \log_3(2x)^{10}\,dx = 10\times\text{"13.3"} = \text{awrt}\,133$ or e.g. $\frac{531}{4}$	B1ft	2.2a
(ii)	$\int_3^9 \log_3 18x\,dx = \int_3^9 \log_3(9\times 2x)\,dx = \int_3^9 2 + \log_3 2x\,dx$ $= [2x]_3^9 + \int_3^9 \log_3 2x\,dx = 18 - 6 + \int_3^9 \log_3 2x\,dx = \ldots$	M1	3.1a
	Awrt 25.3 or $\frac{1011}{40}$	A1ft	1.1b
		(3)	
			(6 marks)

Notes:

(a)

B1: States or uses $h = 1.5$

M1: A full attempt at the trapezium rule.

Look for $\frac{\text{their } h}{2}\{1.63 + 2.63 + 2\times(2 + 2.26 + 2.46)\}$ but condone copying slips

Note that $\frac{\text{their } h}{2}1.63 + 2.63 + 2\times(2 + 2.26 + 2.46)$ scores M0 unless the missing brackets are recovered or implied by their answer. You may need to check.

Allow this mark if they add the areas of individual trapezia e.g.

$\frac{\text{their } h}{2}\{1.63 + 2\} + \frac{\text{their } h}{2}\{2 + 2.26\} + \frac{\text{their } h}{2}\{2.26 + 2.46\} + \frac{\text{their } h}{2}\{2.46 + 2.63\}$

Condone copying slips but must be a complete method using all the trapezia.

A1: awrt 13.3 (Note full accuracy is 13.275) or exact equivalent.

Note that the calculator answer is 13.324 so you must see correct working to award awrt 13.3

Use of $h = -1.5$ leading to a negative area can score B1M1A0 but allow full marks if then stated as positive.

(b)(i)

B1ft: Deduces that $\int_3^9 \log_3(2x)^{10}\,dx = 10\times\text{"13.3"} = \text{awrt } 133$

FT on their 13.3 look for 3sf accuracy but follow through on e.g. their rounded answer to part (a) so if 13 was their answer to part (a) then allow 130 here **following a correct method**.

A correct method must be seen here but a minimum is e.g. 10 × "13.3" = "133"

Note that $\int_3^9 \log_3(2x)^{10}\,dx = 133.2414316\ldots$ so a correct method must be seen to award marks.

Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the <u>answer</u> to part (a).

M1: Shows correct log work to relate the given question to part (a)

Must reach as far as e.g. $\left[2x\right]_3^9 + \int_3^9 \log_3 2x \, \mathrm{d}x = ...$ with correct use of limits on $\left[2x\right]_3^9$ which may be implied or equivalent work e.g. finds the area of the rectangle as 2×6

A1ft: **Correct working** followed by awrt 25.3 but ft on their 13.3 so allow for 12 + their answer to part (a) **following correct work** as shown.

Note that $\int_3^9 \log_3 18x \, \mathrm{d}x = 25.32414...$ **so a correct method must be seen to award marks**.

Some examples of an acceptable method are:

$$\int_3^9 \log_3 18x \, \mathrm{d}x = \int_3^9 \log_3 \left(9 \times 2x\right) \mathrm{d}x = \int_3^9 2 + \log_3 2x \, \mathrm{d}x = 6 \times 2 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, \mathrm{d}x = \int_3^9 \log_3 \left(9 \times 2x\right) \mathrm{d}x = \int_3^9 2 + \log_3 2x \, \mathrm{d}x = 12 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, \mathrm{d}x = \int_3^9 \log_3 \left(9 \times 2x\right) \mathrm{d}x = \int_3^9 2 + \log_3 2x \, \mathrm{d}x = \left[2x\right]_3^9 + \int_3^9 \log_3 2x \, \mathrm{d}x = 25.3$$

BUT just $12 + "13.3" = 25.3$ **scores M0**

Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the <u>answer</u> to part (a).

8	$y=\dfrac{(x-2)(x-4)}{4\sqrt{x}}=\dfrac{x^2-6x+8}{4\sqrt{x}}=\dfrac{1}{4}x^{\frac{3}{2}}-\dfrac{3}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}$	M1 A1	1.1b 1.1b
	$\int \dfrac{1}{4}x^{\frac{3}{2}}-\dfrac{3}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}\mathrm{d}x=\dfrac{1}{10}x^{\frac{5}{2}}-x^{\frac{3}{2}}+4x^{\frac{1}{2}}(+c)$	dM1 A1	3.1a 1.1b
	Deduces limits of integral are 2 and 4 and applies to their $\dfrac{1}{10}x^{\frac{5}{2}}-x^{\frac{3}{2}}+4x^{\frac{1}{2}}$	M1	2.2a
	$\left(\dfrac{32}{10}-8+8\right)-\left(\dfrac{2}{5}\sqrt{2}-2\sqrt{2}+4\sqrt{2}\right)=\dfrac{16}{5}-\dfrac{12}{5}\sqrt{2}$ Area R $=\dfrac{12}{5}\sqrt{2}-\dfrac{16}{5}$ $\left(\text{or } \dfrac{16}{5}-\dfrac{12}{5}\sqrt{2}\right)$	A1	2.1
		(6)	
			(6 marks)

Notes:

M1: Correct attempt to write $\dfrac{(x-2)(x-4)}{4\sqrt{x}}$ as a sum of terms with **indices**.

Look for at least two different terms with the correct index e.g. two of $x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$, $x^{-\frac{1}{2}}$ which have come from the correct places.

The correct indices may be implied later when e.g. $\sqrt{x}$ becomes $x^{\frac{1}{2}}$ or $\dfrac{1}{\sqrt{x}}$ becomes $x^{-\frac{1}{2}}$

A1: $\dfrac{1}{4}x^{\frac{3}{2}}-\dfrac{3}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}$ which can be left unsimplified e.g. $\dfrac{1}{4}x^{2-\frac{1}{2}}-\dfrac{1}{2}x^{\frac{1}{2}}-x^{\frac{1}{2}}+2x^{-\frac{1}{2}}$

or as e.g. $\dfrac{1}{4}\left(x^{\frac{3}{2}}-6x^{\frac{1}{2}}+8x^{-\frac{1}{2}}\right)$

The correct indices may be implied later when e.g. $\sqrt{x}$ becomes $x^{\frac{1}{2}}$ or $\dfrac{1}{\sqrt{x}}$ becomes $x^{-\frac{1}{2}}$

dM1: Integrates $x^n \to x^{n+1}$ for at least 2 correct indices

i.e. at least 2 of $x^{\frac{3}{2}} \to x^{\frac{5}{2}}$, $x^{\frac{1}{2}} \to x^{\frac{3}{2}}$, $x^{-\frac{1}{2}} \to x^{\frac{1}{2}}$

It is dependent upon the first M so at least two terms must have had a correct index.

A1: $\dfrac{1}{10}x^{\frac{5}{2}}-x^{\frac{3}{2}}+4x^{\frac{1}{2}}(+c)$. Allow unsimplified e.g. $\dfrac{1}{4}\times\dfrac{2}{5}x^{\frac{3}{2}+1}-\dfrac{1}{2}\times\dfrac{2}{3}x^{\frac{1}{2}+1}-\dfrac{2}{3}x^{\frac{1}{2}+1}+2\times 2x^{\frac{1}{2}}$

or as e.g. $\dfrac{1}{4}\left(\dfrac{2}{5}x^{\frac{5}{2}}-4x^{\frac{3}{2}}+16x^{\frac{1}{2}}\right)(+c)$.

M1: Substitutes the limits 4 and 2 to their $\dfrac{1}{10}x^{\frac{5}{2}}-x^{\frac{3}{2}}+4x^{\frac{1}{2}}$ and subtracts either way round.

There is no requirement to evaluate but 4 and 2 must be substituted either way round with evidence of subtraction, condoning omission of brackets.

E.g. condone $\dfrac{1}{10}\times 4^{\frac{5}{2}}-4^{\frac{3}{2}}+4\times 4^{\frac{1}{2}}-\dfrac{1}{10}\times 2^{\frac{5}{2}}-2^{\frac{3}{2}}+4\times 2^{\frac{1}{2}}$

This is an independent mark but the limits must be applied to an expression that is not *y* so they may even have differentiated.

Award this mark once one of these forms is reached and isw

See overleaf for integration by parts and integration by substitution.

$\int \frac{(x-2)(x-4)}{4\sqrt{x}}\,dx = \int \frac{1}{4}(x-2)(x-4)x^{-\frac{1}{2}}\,dx = \frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \int \frac{1}{2}(2x-6)x^{\frac{1}{2}}\,dx$	M1 A1	1.1b 1.1b
$\frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \int \frac{1}{2}(2x-6)x^{\frac{1}{2}}\,dx = \frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \int x^{\frac{3}{2}} - 3x^{\frac{1}{2}}dx$ $= \frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}}$ Or e.g. $= \frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}(2x-6) + \frac{4}{15}x^{\frac{5}{2}}$	dM1 A1	3.1a 1.1b
Deduces limits of integral are 2 and 4 and applies to their $\frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}(2x-6) + \frac{4}{15}x^{\frac{5}{2}}$	M1	2.2a
$0 - \frac{16}{3} + \frac{128}{15} - \left(0 + \frac{4}{3}\sqrt{2} + \frac{16}{15}\sqrt{2}\right)$ Area $R = \frac{12}{5}\sqrt{2} - \frac{16}{5}$ $\left(\text{or } \frac{16}{5} - \frac{12}{5}\sqrt{2}\right)$	A1	2.1
	(6)	

Notes:

M1: Applies integration by parts and reaches the form $\alpha(x-2)(x-4)x^{\frac{1}{2}} \pm \int (px+q)x^{\frac{1}{2}}\,dx$ $\alpha, p \neq 0$

oe e.g. $\alpha\left(x^2 - 6x + 8\right)x^{\frac{1}{2}} \pm \int (px+q)x^{\frac{1}{2}}\,dx$ $\alpha, p \neq 0$

A1: Correct first application of parts in any form

dM1: Attempts their $\int (px+q)x^{\frac{1}{2}}\,dx$ by expanding and integrating or may attempt parts again.

E.g. $\int (2x-6)x^{\frac{1}{2}}\,dx = \int \left(2x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right)dx = \ldots$ or e.g. $\int (2x-6)x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}(2x-6) - \frac{4}{3}\int x^{\frac{3}{2}}\,dx$

If they expand then at least one term requires $x^n \to x^{n+1}$ or if parts is attempted again, the structure must be correct.

A1: Fully correct integration in any form

M1: Substitutes the limits 4 and 2 to their $= \frac{1}{2}(x-2)(x-4)x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}}$ and subtracts

either way round. There is no requirement to evaluate but 4 and 2 must be substituted either way round with evidence of subtraction, condoning omission of brackets.

E.g. condone $0 - \frac{16}{3} + \frac{128}{15} - 0 + \frac{4}{3}\sqrt{2} + \frac{16}{15}\sqrt{2}$

This is an independent mark but the limits must be applied to a "changed" function.

A1: Correct working shown leading to $\frac{12}{5}\sqrt{2} - \frac{16}{5}$ but also allow $\frac{16}{5} - \frac{12}{5}\sqrt{2}$ or exact equivalents

$u=\sqrt{x}\left(x=u^2\right)\Rightarrow\int\frac{(x-2)(x-4)}{4\sqrt{x}}\mathrm{d}x=\int\frac{\left(u^2-2\right)\left(u^2-4\right)}{4u}\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ $=\int\frac{\left(u^2-2\right)\left(u^2-4\right)}{4u}2u\mathrm{d}u$	M1 A1	1.1b 1.1b
$=\frac{1}{2}\int\left(u^4-6u^2+8\right)\mathrm{d}u=\frac{1}{2}\left(\frac{u^5}{5}-\frac{6u^3}{3}+8u\right)(+c)$	dM1 A1	3.1a 1.1b
Deduces limits of integral are $\sqrt{2}$ and 2 and applies to their $\frac{1}{2}\left(\frac{u^5}{5}-\frac{6u^3}{3}+8u\right)$	M1	2.2a
$\frac{1}{2}\left(\frac{32}{5}-16+16-\left(\frac{4\sqrt{2}}{5}-4\sqrt{2}+8\sqrt{2}\right)\right)$ Area R $=\frac{12}{5}\sqrt{2}-\frac{16}{5}$ $\left(\text{or }\frac{16}{5}-\frac{12}{5}\sqrt{2}\right)$	A1	2.1
	(6)	

Notes:

M1: Applies the substitution e.g. $u=\sqrt{x}$ and attempts $k\int\frac{\left(u^2-2\right)\left(u^2-4\right)}{u}\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$

A1: Fully correct integral in terms of u in any form e.g. $\frac{1}{2}\int\left(u^2-2\right)\left(u^2-4\right)\mathrm{d}u$

dM1: Expands the bracket and integrates $u^n\to u^{n+1}$ for at least 2 correct indices

i.e. at least 2 of $u^4\to u^5$, $u^2\to u^3$, $k\to ku$

A1: $\frac{1}{2}\left(\frac{u^5}{5}-\frac{6u^3}{3}+8u\right)(+c)$. Allow unsimplified.

M1: Substitutes the limits 2 and $\sqrt{2}$ to their $\frac{1}{2}\left(\frac{u^5}{5}-\frac{6u^3}{3}+8u\right)$ and subtracts either way round.

There is no requirement to evaluate but 2 and $\sqrt{2}$ must be substituted either way round with evidence of subtraction, condoning omission of brackets.

E.g. condone $\frac{1}{2}\left(\frac{32}{5}-16+16-\frac{4\sqrt{2}}{5}-4\sqrt{2}+8\sqrt{2}\right)$

Alternatively reverses the substitution and applies the limits 4 and 2 with the same conditions.

A1: Correct working shown leading to $\frac{12}{5}\sqrt{2}-\frac{16}{5}$ but also allow $\frac{16}{5}-\frac{12}{5}\sqrt{2}$ or exact equivalents

Award this mark once one of these forms is reached and isw.

There may be other substitutions seen and the same marking principles apply.

7(a)	$y = x^3 - 10x^2 + 27x - 23 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 20x + 27$	B1	1.1b
	$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=5} = 3\times 5^2 - 20\times 5 + 27 \left(= 2\right)$	M1	1.1b
	$y + 13 = 2\left(x - 5\right)$	M1	2.1
	$y = 2x - 23$	A1	1.1b
		(4)	
(b)	Both C and l pass through $(0, -23)$ and so C meets l again on the y-axis	B1	2.2a
		(1)	
(c)	$\pm\int \left(x^3 - 10x^2 + 27x - 23 - \left(2x - 23\right)\right) \mathrm{d}x$ $= \pm\left(\dfrac{x^4}{4} - \dfrac{10}{3}x^3 + \dfrac{25}{2}x^2\right)$	M1 A1ft	1.1b 1.1b
	$\left[\dfrac{x^4}{4} - \dfrac{10}{3}x^3 + \dfrac{25}{2}x^2\right]_0^5$ $= \left(\dfrac{625}{4} - \dfrac{1250}{3} + \dfrac{625}{2}\right)(-0)$	dM1	2.1
	$= \dfrac{625}{12}$	A1	1.1b
		(4)	
	(c) Alternative:		
	$\pm\int \left(x^3 - 10x^2 + 27x - 23\right) \mathrm{d}x$ $= \pm\left(\dfrac{x^4}{4} - \dfrac{10}{3}x^3 + \dfrac{27}{2}x^2 - 23x\right)$	M1 A1	1.1b 1.1b
	$\left[\dfrac{x^4}{4} - \dfrac{10}{3}x^3 + \dfrac{27}{2}x^2 - 23x\right]_0^5 + \dfrac{1}{2}\times 5\left(23 + 13\right)$ $= -\dfrac{455}{12} + 90$	dM1	2.1
	$= \dfrac{625}{12}$	A1	1.1b
			(9 marks)

Notes

(a)

B1: Correct derivative

M1: Substitutes $x = 5$ into their derivative. This may be implied by their value for $\frac{\mathrm{d}y}{\mathrm{d}x}$

M1: Fully correct straight line method using $(5, -13)$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 5$

A1: cao. Must see the full equation in the required form.

(b)

B1: Makes a suitable deduction.

Alternative via equating l and C and factorising e.g.

$$x^3 - 10x^2 + 27x - 23 = 2x - 23$$

$$x^3 - 10x^2 + 25x = 0$$

$$x\left(x^2 - 10x + 25\right) = 0 \Rightarrow x = 0$$

So they meet on the y-axis

(c)

M1: For an attempt to integrate $x^n \to x^{n+1}$ for $\pm$" $C - l$"

A1ft: Correct integration in any form which may be simplified or unsimplified. (follow through their equation from (a))

If they attempt as 2 separate integrals e.g. $\int \left(x^3 - 10x^2 + 27x - 23\right) \mathrm{d}x - \int \left(2x - 23\right) \mathrm{d}x$ then award this mark for the correct integration of the curve as in the alternative.

If they combine the curve with the line first then the subsequent integration must be correct or a correct ft for their line and allow for $\pm$" $C - l$"

dM1: Fully correct strategy for the area. Award for use of 5 as the limit and condone the omission of the "– 0". **Depends on the first method mark.**

A1: Correct exact value

Alternative:

M1: For an attempt to integrate $x^n \to x^{n+1}$ for $\pm C$

A1: Correct integration for $\pm C$

dM1: Fully correct strategy for the area e.g. correctly attempts the area of the trapezium and subtracts the area enclosed between the curve and the x-axis. Need to see the use of 5 as the limit condoning the omission of the "– 0" **and** a correct attempt at the trapezium **and** the subtraction.

May see the trapezium area attempted as $\int \left(2x - 23\right) \mathrm{d}x$ in which case the integration and use of the limits needs to be correct or correct follow through for their straight line equation.

Depends on the first method mark.

A1: Correct exact value

Note if they do $l - C$ rather than $C - l$ and the working is otherwise correct allow full marks if their final answer is given as a positive value. E.g. correct work with $l - C$ leading to $-\frac{625}{12}$ and then e.g. hence area is $\frac{625}{12}$ is acceptable for full marks.

If the answer is left as $-\frac{625}{12}$ then score A0

Question	Scheme	Marks	AOs

12(a)	$u=1+\sqrt{x} \Rightarrow x=(u-1)^2 \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u}=2(u-1)$ or $u=1+\sqrt{x} \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\displaystyle\int \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int \frac{(u-1)^2}{u}2(u-1)\mathrm{d}u$ or $\displaystyle\int \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int \frac{x}{u}\times 2x^{\frac{1}{2}}\mathrm{d}u=\int \frac{2x^{\frac{3}{2}}}{u}\mathrm{d}u=\int \frac{2(u-1)^3}{u}\mathrm{d}u$	M1	2.1
	$\displaystyle\int_0^{16} \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int_1^5 \frac{2(u-1)^3}{u}\mathrm{d}u$	A1	1.1b
		(3)	
(b)	$\displaystyle 2\int \frac{u^3-3u^2+3u-1}{u}\mathrm{d}u=2\int \left(u^2-3u+3-\frac{1}{u}\right)\mathrm{d}u=\ldots$	M1	3.1a
	$\displaystyle =(2)\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]$	A1	1.1b
	$\displaystyle =2\left[\frac{5^3}{3}-\frac{3(5)^2}{2}+3(5)-\ln 5-\left(\frac{1}{3}-\frac{3}{2}+3-\ln 1\right)\right]$	dM1	2.1
	$\displaystyle =\frac{104}{3}-2\ln 5$	A1	1.1b
		(4)	
			(7 marks)

Notes

(a)

B1: Correct expression for $\dfrac{\mathrm{d}x}{\mathrm{d}u}$ or $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ $(\text{or } u')$ or dx in terms of du or du in terms of dx

M1: Complete method using the given substitution.

This needs to be a correct method for their $\dfrac{\mathrm{d}x}{\mathrm{d}u}$ or $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see:

$\dfrac{\mathrm{d}x}{\mathrm{d}u}=\mathrm{f}(u)\rightarrow \displaystyle\int \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int \frac{(u-1)^2}{u}\mathrm{f}(u)\mathrm{d}u$

$\dfrac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{g}(x)\rightarrow \displaystyle\int \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int \frac{x}{u}\times\frac{\mathrm{d}u}{\mathrm{g}(x)}=\int \mathrm{h}(u)\mathrm{d}u$. In this case you can condone slips with coefficients e.g. allow $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2}x^{-\frac{1}{2}}\rightarrow \displaystyle\int \frac{x}{1+\sqrt{x}}\mathrm{d}x=\int \frac{x}{u}\times\frac{x^{\frac{1}{2}}}{2}\mathrm{d}u=\int \mathrm{h}(u)\mathrm{d}u$

but not $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow \int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u} \times \frac{x^{\frac{1}{2}}}{2}du = \int h(u)du$

A1: All correct with correct limits and no errors. The "du" must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately.

(b)

M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3, ku^2, ku, $k\ln u$

A1: Correct integration. This mark can be scored with the "2" still outside the integral or even if it has been omitted. But if the "2" has been combined with the integrand, the integration must be correct.

dM1: Completes the process by applying their "changed" limits and subtracts the right way round **Depends on the first method mark.**

A1: Cao (Allow equivalents for $\frac{104}{3}$ e.g. $\frac{208}{6}$)

14(a)	$\frac{dV}{dt} = 0.48 - 0.1h$	B1	3.1b
	$V = 24h \Rightarrow \frac{dV}{dh} = 24$ or $\frac{dh}{dV} = \frac{1}{24}$	B1	3.1b
	$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{0.48 - 0.1h}{24}$ or e.g. $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.48 - 0.1h = 24\frac{dh}{dt}$	M1	2.1
	$1200\frac{dh}{dt} = 24 - 5h$*	A1*	1.1b
		(4)	
(b)	$1200\frac{dh}{dt} = 24 - 5h \Rightarrow \int \frac{1200}{24-5h}dh = \int dt$ $\Rightarrow e.g.\ \alpha \ln(24-5h) = t(+c)$ oe or $1200\frac{dh}{dt} = 24 - 5h \Rightarrow \frac{dt}{dh} = \frac{1200}{24-5h}$ $\Rightarrow e.g.\ t(+c) = \alpha \ln(24-5h)$ oe	M1	3.1a
	$t = -240\ln(24-5h)(+c)$ oe	A1	1.1b
	$t = 0, h = 2 \Rightarrow 0 = -240\ln(24-10) + c \Rightarrow c = ...(240\ln 14)$	M1	3.4
	$t = 240\ln(14) - 240\ln(24-5h)$	A1	1.1b
	$t = 240\ln\frac{14}{24-5h} \Rightarrow \frac{t}{240} = \ln\frac{14}{24-5h} \Rightarrow e^{\frac{t}{240}} = \frac{14}{24-5h}$ $\Rightarrow 14e^{-\frac{t}{240}} = 24 - 5h \Rightarrow h = ...$	ddM1	2.1
	$h = 4.8 - 2.8e^{-\frac{t}{240}}$ oe e.g. $h = \frac{24}{5} - \frac{14}{5}e^{-\frac{t}{240}}$	A1	3.3
		(6)	
(c)	**Examples:** • As $t \to \infty, e^{-\frac{t}{240}} \to 0$ • When $h > 4.8$, $\frac{dV}{dt} < 0$ • Flow in = flow out at max h so $0.1h = 4.8 \rightarrow h = 4.8$ • As $e^{-\frac{t}{240}} > 0,\ h < 4.8$ • $h = 5 \Rightarrow \frac{dV}{dt} = -0.02$ or $\frac{dh}{dt} = -\frac{1}{1200}$ • $\frac{dh}{dt} = 0 \Rightarrow h = 4.8$ • $h = 5 \Rightarrow 4.8 - 2.8e^{-\frac{t}{240}} = 5 \Rightarrow e^{-\frac{t}{240}} < 0$	M1	3.1b
	• The limit for h (according to the model) is 4.8m and the tank is 5m high so the tank will never become full • If $h = 5$ the tank would be emptying so can never be full • The equation can't be solved when $h = 5$	A1	3.2a

(12 marks)

Notes

(a)

B1: Identifies the correct expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$ according to the model

B1: Identifies the correct expression for $\frac{\mathrm{d}V}{\mathrm{d}h}$ according to the model

M1: Applies $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}t}\times\frac{\mathrm{d}h}{\mathrm{d}V}$ or equivalent correct formula with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ which may be implied by their working

A1*: Correct equation obtained with no errors

Note that: $\frac{\mathrm{d}V}{\mathrm{d}t}=0.48-0.1h\Rightarrow\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{0.48-0.1h}{24}\Rightarrow 1200\frac{\mathrm{d}h}{\mathrm{d}t}=24-5h$ * scores

B1B0M0A0. There must be clear evidence where the "24" comes from and evidence of the correct chain rule being applied.

(b)

M1: Adopts a correct strategy by separating the variables correctly or rearranges to obtain $\frac{\mathrm{d}t}{\mathrm{d}h}$ correctly in terms of h **and** integrates to obtain $t=\alpha\ln\left(24-5h\right)\left(+c\right)$ or equivalent (condone missing brackets around the "24 – 5h") and + c not required for this mark.

A1: Correct equation in any form and + c not required. Do not condone missing brackets unless they are implied by subsequent work.

M1: Substitutes t = 0 and h = 2 to find their constant of integration (there must have been some attempt to integrate)

A1: Correct equation in any form

ddM1: Uses fully correct log work to obtain h in terms of t.

This depends on <u>both</u> previous method marks.

A1: Correct equation

Note that the marks may be earned in a different order e.g.:

$$t+c=-240\ln\left(24-5h\right)\Rightarrow-\frac{t}{240}+d=\ln\left(24-5h\right)\Rightarrow A\mathrm{e}^{-\frac{t}{240}}=24-5h$$

$$t=0,h=2\Rightarrow A=14\Rightarrow 14\mathrm{e}^{-\frac{t}{240}}=24-5h\Rightarrow h=4.8-2.8\mathrm{e}^{-\frac{t}{240}}$$

Score as M1 A1 as in main scheme then

M1: Correct work leading to $A\mathrm{e}^{\alpha t}=24-5h$ (must have a constant "A")

A1: $A\mathrm{e}^{-\frac{t}{240}}=24-5h$

ddM1: Uses t = 0, h = 2 in an expression of the form above to find A

A1: $h=4.8-2.8\mathrm{e}^{-\frac{t}{240}}$

(c)

M1: See scheme for some examples

A1: Makes a correct interpretation for their method.

There must be no incorrect working or contradictory statements.

This is not a follow through mark and if their equation in (b) is used it must be correct.

Question	Scheme	Marks	AOs
8	Any equation involving an exponential of the correct form. See notes	M1	3.1b
	$n = Ae^{kt}$ (where A and k are positive constants)	A1	1.1b
		(2)	
			(2 marks)
Notes:			

M1: Any equation of the correct form, involving n and an exponential in t.

So allow for example $n = e^{\pm t}$, $n = Ae^{\pm t}$, $n = Ae^{\pm kt}$ condoning $n = A + Be^{\pm t}$

Condone an intermediate form where n has not been made the subject.

E.g Allow $\ln n = kt + c$ but also condone $\ln n = kt$

A1: E.g. $n = Ae^{kt}$, $n = e^{kt+c}$, $n = e^{kt}e^{c}$ There is no requirement to state that A and k are positive constants

Note that the two constants need to be different.

Mark the final answer so $n = e^{kt+c}$ followed by $n = e^{kt} + e^{c}$ o.e. $n = e^{kt} + A$ such as is M1 A0

.....

You may see solutions that don't include "e".

These are fine so you can include versions of $n = Ak^{t}$ using the same marking criteria as above

FYI $\frac{dn}{dt} = Ak^{t} \times \ln k = \ln k \times n$ so $\frac{dn}{dt} \propto n$

.....

Question	Scheme	Marks	AOs

10 (a)	$x = u^2 + 1 \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$ oe	B1	1.1b
	Full substitution $\int \frac{3\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int \frac{3\times 2u\,\mathrm{d}u}{\left(u^2+1-1\right)(3+2u)}$	M1	1.1b
	Finds correct limits e.g. $p = 2, q = 3$	B1	1.1b
	$= \int \frac{3\times 2\cancel{u}\,\mathrm{d}u}{\cancel{u^2}\,(3+2u)} = \int \frac{6\,\mathrm{d}u}{u(3+2u)}$ *	A1*	2.1
		(4)	
(b)	$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = ..., B = ...$	M1	1.1b
	Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	A1	1.1b
	$\int \frac{6\,\mathrm{d}u}{u(3+2u)} = 2\ln u - 2\ln(3+2u) \qquad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct ln work leading to $k\ln b$ E.g. $(2\ln 3 - 2\ln 9) - (2\ln 2 - 2\ln 7) = 2\ln\frac{7}{6}$	M1	1.1b
	$\ln\frac{49}{36}$	A1	2.1
		(6)	
			(10 marks)
Notes: Mark (a) and (b) together as one complete question			

(a)

B1: $\mathrm{d}x = 2u\mathrm{d}u$ or exact equivalent. E.g. $\frac{\mathrm{d}x}{\mathrm{d}u} = 2u$, $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

M1: Attempts a full substitution of $x = u^2 + 1$, including $\mathrm{d}x \to ...u\mathrm{d}u$ to form an integrand in terms of *u*. Condone slips but there should be an attempt to use the correct substitution on the denominator.

B1: Finds correct limits either states $p = 2, q = 3$ or sight of embedded values as limits to the integral

A1*: Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression ignoring limits. So B1, M1, B0, A1 is possible if the limits are incorrect, omitted or left as 5 and 10.

(b)

M1: Uses correct form of PF leading to values of *A* and *B*.

A1: Correct PF $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$ (Not scored for just the correct values of *A* and *B*)

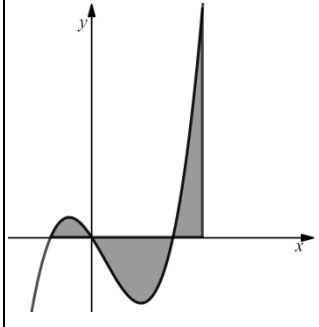
dM1: This is an overall problem solving mark. It is for using the correct PF form and integrating using lns. Look for $P\ln u + Q\ln(3+2u)$

A1ft: Correct integration for their $\frac{A}{u} + \frac{B}{3+2u} \to A\ln u + \frac{B}{2}\ln(3+2u)$ with or without modulus signs

M1: Uses their 2 and 3 as limits, with at least one correct application of the addition law or subtraction law leading to the form $k\ln b$ or $\ln a$. PF's must have been attempted. Condone bracketing slips. Alternatively changing the *u*'s back to *x*'s and use limits of 5 and 10.

A1: Proceeds to $\ln\frac{49}{36}$. Answers without working please send to review.

Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
8 (a)	$y = x(x+2)(x-4) = x^3 - 2x^2 - 8x$	B1	1.1b
	$\int x^3 - 2x^2 - 8x \, dx \rightarrow \frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2$	M1	1.1b
	Attempts area using the correct strategy $\int_{-2}^{0} y \, dx$	dM1	2.2a
	$\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2\right]_{-2}^{0} = (0) - \left(4 - \frac{-16}{3} - 16\right) = \frac{20}{3}$ *	A1*	2.1
		(4)	
(b)	For setting 'their' $\frac{1}{4}b^4 - \frac{2}{3}b^3 - 4b^2 = \pm\frac{20}{3}$	M1	1.1b
	For correctly deducing that $3b^4 - 8b^3 - 48b^2 + 80 = 0$	A1	2.2a
	Attempts to factorise $3b^4 - 8b^3 - 48b^2 \pm 80 = (b+2)(b+2)(3b^2 ... b ... 20)$	M1	1.1b
	Achieves $(b+2)^2\left(3b^2 - 20b + 20\right) = 0$ with no errors	A1*	2.1
		(4)	
(c)	 States that between $x = -2$ and $x = 5.442$ the area above the x-axis = area below the x-axis	B1 B1	1.1b 2.4
		(2)	
			(10 marks)

(a)

B1: Expands $x(x+2)(x-4)$ to $x^3 - 2x^2 - 8x$ (They may be in a different order)

M1: Correct attempt at integration of their cubic seen in at least two terms.

Look for an expansion to a cubic and $x^n \rightarrow x^{n+1}$ seen at least twice

dM1: For a correct strategy to find the area of R_1

It is dependent upon the previous M and requires a substitution of -2 into $\pm$ their integrated function.

The limit of 0 may not be seen. Condone $\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2\right]_{-2}^{0} = \frac{20}{3}$ oe for this mark

A1*: For a rigorous argument leading to area of $R_1 = \frac{20}{3}$ For this to be awarded the integration must be correct and the limits must be the correct way around and embedded or calculated values must be seen.

Eg. Look for $-\left(4+\frac{16}{3}-16\right)$ or $-\left(\frac{1}{4}(-2)^4-\frac{2}{3}(-2)^3-4(-2)^2\right)$ oe before you see the $\frac{20}{3}$

Note: It is possible to do this integration by parts.

(b)

M1: For setting their $\frac{1}{4}b^4-\frac{2}{3}b^3-4b^2=\pm\frac{20}{3}$ or $\left[\frac{1}{4}x^4-\frac{2}{3}x^3-4x^2\right]_{-2}^{b}=0$

A1: Deduces that $3b^4-8b^3-48b^2+80=0$. Terms may be in a different order but expect integer coefficients.

It must have followed $\frac{1}{4}b^4-\frac{2}{3}b^3-4b^2=-\frac{20}{3}$ oe.

Do not award this mark for $\frac{1}{4}b^4-\frac{2}{3}b^3-4b^2+\frac{20}{3}=0$ unless they attempt the second part of this question by expansion and then divide the resulting expanded expression by 12

M1: Attempts to factorise $3b^4-8b^3-48b^2\pm80=(b+2)(b+2)(3b^2...b...20)$ via repeated division or inspection. FYI $3b^4-8b^3-48b^2+80=(b+2)(3b^3-14b^2-20b+40)$ Allow an attempt via inspection $3b^4-8b^3-48b^2\pm80=(b^2+4b+4)(3b^2...b...20)$ but do not allow candidates to just write out $3b^4-8b^3-48b^2\pm80=(b+2)^2(3b^2-20b+20)$ which is really just copying out the given answer.

Alternatively attempts to expand $(b+2)^2(3b^2-20b+20)$ achieving terms of a quartic expression

A1*: Correctly reaches $(b+2)^2(3b^2-20b+20)=0$ with no errors and must have $=0$

In the alternative obtains both equations in the same form **and states that they are same**. Allow ✓ QED etc here.

(c) Please watch for candidates who answer this on Figure 2 which is fine

B1: Sketches the curve and a vertical line to the right of 4 (x = 5.442 may not be labelled.)

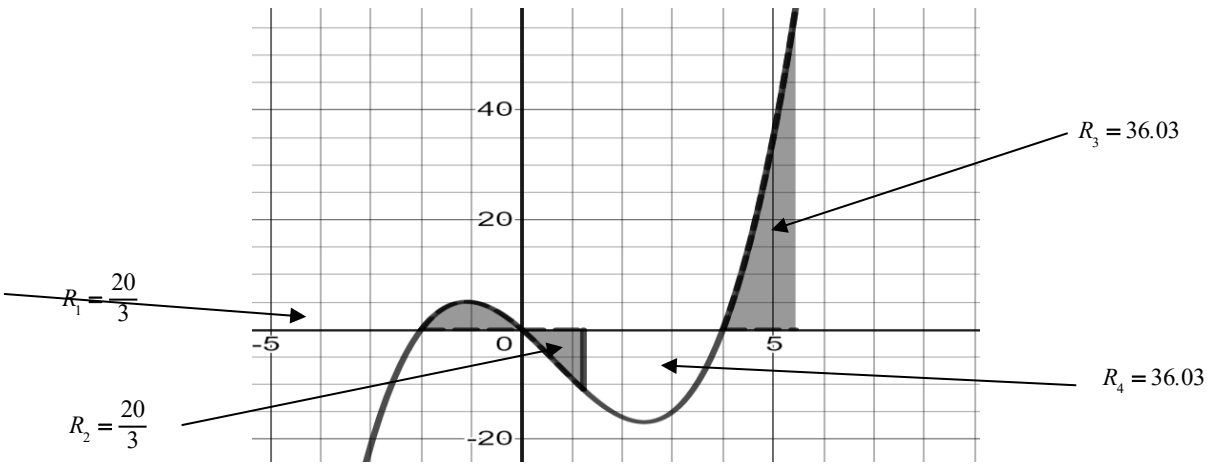
B1: Explains that (between $x=-2$ and $x=5.442$) the area above the x-axis = area below the x -axis with appropriate areas shaded or labelled**.**

Alternatively states that the area between 1.225 and 4 is the same as the area between 4 and 5.442

Another correct statement is that the net area between 0 and 5.442 is $-\frac{20}{3}$

Look carefully at what is written. There are many correct statements/ deductions.

Eg. " (area between 0 and 4) - (area between 4 and 5.442) = 20/3". Diagram below for your information.



Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
7 (a)	$\int \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(3x-k)$	M1 A1	1.1a 1.1b
	$\int_k^{3k} \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(9k-k) - \frac{2}{3}\ln(3k-k)$	dM1	1.1b
	$= \frac{2}{3}\ln\left(\frac{8\cancel{k}}{2\cancel{k}}\right) = \frac{2}{3}\ln 4$ oe	A1	2.1
		(4)	
(b)	$\int \frac{2}{(2x-k)^2}\,dx = -\frac{1}{(2x-k)}$	M1	1.1b
	$\int_k^{2k} \frac{2}{(2x-k)^2}\,dx = -\frac{1}{(4k-k)} + \frac{1}{(2k-k)}$	dM1	1.1b
	$= \frac{2}{3k} \quad \left(\propto \frac{1}{k}\right)$	A1	2.1
		(3)	
			(7 marks)

(a)

M1: $\int \frac{2}{(3x-k)}\,dx = A\ln(3x-k)$ Condone a missing bracket

A1: $\int \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(3x-k)$

Allow recovery from a missing bracket if in subsequent work $A\ln 9k-k \rightarrow A\ln 8k$

dM1: For substituting k and $3k$ into their $A\ln(3x-k)$ and subtracting either way around

A1: Uses correct ln work and notation to show that I $= \frac{2}{3}\ln\left(\frac{8}{2}\right)$ or $\frac{2}{3}\ln 4$ oe (ie independent of k)

(b)

M1: $\int \frac{2}{(2x-k)^2}\,dx = \frac{C}{(2x-k)}$

dM1: For substituting k and $2k$ into their $\frac{C}{(2x-k)}$ and subtracting

A1: Shows that it is inversely proportional to k Eg proceeds to the answer is of the form A/k with $A = \frac{2}{3}$

There is no need to perform the whole calculation. Accept from $-\frac{1}{(3k)} + \frac{1}{(k)} = \left(-\frac{1}{3}+1\right)\times\frac{1}{k} \propto \frac{1}{k}$

If the calculation is performed it must be correct.

Do not isw here. They should know when they have an expression that is inversely proportional to *k*.

You may see substitution used but the mark is scored for the same result. See below

$u = 2x-k \rightarrow \left[\frac{C}{\quad}\right]$ for M1 with limits $3k$ and k used for dM1

Question	Scheme	Marks	AOs

13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x$ Award for • Using a valid substitution u =..., changing the terms to u's • integrating and using appropriate limts .	M1	3.1a
	Substitution $u=\sqrt{x+2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=2u$ oe	Substitution $u=x+2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=1$ oe	B1	1.1b
	$\int 2x\sqrt{x+2}\,\mathrm{d}x$ $=\int A\left(u^2 \pm 2\right)u^2\mathrm{d}u$	$\int 2x\sqrt{x+2}\,\mathrm{d}x$ $=\int A\left(u \pm 2\right)\sqrt{u}\mathrm{d}u$	M1	1.1b
	$=Pu^5 \pm Qu^3$	$=Su^{\frac{5}{2}} \pm Tu^{\frac{3}{2}}$	dM1	2.1
	$=\frac{4}{5}u^5-\frac{8}{3}u^3$	$=\frac{4}{5}u^{\frac{5}{2}}-\frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$=\frac{32}{15}\left(2+\sqrt{2}\right)$ *		A1*	2.1
			(7)	
				(7 marks)

M1: For attempting to integrate using substitution. Look for
- terms and limits changed to u's. Condone slips and errors/omissions on changing $\mathrm{d}x \rightarrow \mathrm{d}u$
- attempted multiplication of terms and raising of at least one power of u by one. Condone slips
- Use of at least the top correct limit. For instance if they go back to x's the limit is 2

B1: For substitution it is for giving the substitution and stating a correct $\frac{\mathrm{d}x}{\mathrm{d}u}$

Eg, $u=\sqrt{x+2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=2u$ or equivalent such as $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x+2}}$

M1: It is for attempting to get all aspects of the integral in terms of 'u'.
All terms must be attempted including the dx. You are only condoning slips on signs and coefficients
dM1: It is for using a correct method of expanding and integrating each term (seen at least once) . It is dependent upon the previous M
A1: Correct answer in x or u See scheme
ddM1: Dependent upon the previous M, it is for using the correct limits for their integral, **the correct way around**

A1*: Proceeds correctly to $=\frac{32}{15}\left(2+\sqrt{2}\right)$. **Note that this is a given answer**

There must be at one least correct intermediate line between $\left[\frac{4}{5}u^5-\frac{8}{3}u^3\right]_{\sqrt{2}}^{2}$ and $\frac{32}{15}\left(2+\sqrt{2}\right)$

Alt			
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x$ Award for • using by parts the correct way around • and using limits	M1	3.1a
	$\int\left(\sqrt{x+2}\right)\mathrm{d}x=\frac{2}{3}(x+2)^{\frac{3}{2}}$	B1	1.1b
	$\int 2x\sqrt{x+2}\,\mathrm{d}x=Ax(x+2)^{\frac{3}{2}}-\int B(x+2)^{\frac{3}{2}}(\mathrm{d}x)$	M1	1.1b
	$=Ax(x+2)^{\frac{3}{2}}-C(x+2)^{\frac{5}{2}}$	dM1	2.1
	$=\frac{4}{3}x(x+2)^{\frac{3}{2}}-\frac{8}{15}(x+2)^{\frac{5}{2}}$	A1	1.1b
	Uses limits 2 and 0 the correct way around	ddM1	1.1b
	$=\frac{32}{15}\left(2+\sqrt{2}\right)$	A1*	2.1
		(7)	

M1: For attempting using by parts to solve It is a problem- solving mark and all elements do not have to be correct.

- the formula applied the correct way around. You may condone incorrect attempts at integrating $\sqrt{x+2}$ for this problem solving mark
- further integration, again, this may not be correct, and the use of at least the top limit of 2

B1: For $\int\left(\sqrt{x+2}\right)\mathrm{d}x=\frac{2}{3}(x+2)^{\frac{3}{2}}$ oe May be awarded $\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x\to x^2\times\frac{2(x+2)^{\frac{3}{2}}}{3}$

M1: For integration by parts the right way around. Award for $Ax(x+2)^{\frac{3}{2}}-\int B(x+2)^{\frac{3}{2}}(\mathrm{d}x)$

dM1: For integrating a second time. Award for $Ax(x+2)^{\frac{3}{2}}-C(x+2)^{\frac{5}{2}}$

A1: $\frac{4}{3}x(x+2)^{\frac{3}{2}}-\frac{8}{15}(x+2)^{\frac{5}{2}}$ which may be un simplified

ddM1: Dependent upon the previous M, it is for using the limits 2 and 0 the **correct way around**

A1*: Proceeds to $=\frac{32}{15}\left(2+\sqrt{2}\right)$. **Note that this is a given answer**.

At least one correct intermediate line must be seen. (See substitution). You would condone missing d*x*'s

Question	Scheme	Marks	AOs

10 (a)	$\frac{dr}{dt} \propto \pm\frac{1}{r^2}$ or $\frac{dr}{dt} = \pm\frac{k}{r^2}$ (for k or a numerical k)	M1	3.3
	$\int r^2\,dr = \int \pm k\,dt \Rightarrow \ldots$ (for k or a numerical k)	M1	2.1
	$\frac{1}{3}r^3 = \pm kt \ \{+c\}$	A1	1.1b
	$t=0, r=5$ and $t=4, r=3$ gives $\frac{1}{3}r^3 = -\frac{49}{6}t + \frac{125}{3}$, where r, in mm, is the radius {of the mint} and t, in minutes, is the time from when it {the mint} was placed in the mouth $t=0, r=5$ and $t=240, r=3$ gives $\frac{1}{3}r^3 = -\frac{49}{360}t + \frac{125}{3}$, where r, in mm, is the radius {of the mint} and t, in seconds, is the time from when it {the mint} was placed in the mouth	M1 A1	3.1a 1.1b
		(5)	
(b)	$r=0 \Rightarrow 0 = -\frac{49}{6}t + \frac{125}{3} \Rightarrow 0 = -49t + 250 \Rightarrow t = \ldots$	M1	3.4
	time = 5 minutes 6 seconds	A1	1.1b
		(2)	
(c)	Suggests a suitable limitation of the model. E.g. • Model does not consider how the mint is sucked • Model does not consider whether the mint is bitten • Model is limited for times up to 5 minutes 6 seconds, o.e. • Not valid for times greater than 5 minutes 6 seconds, o.e. • Mint may not retain the shape of a sphere (or have uniform radius) as it is being sucked • The model indicates that the radius of the mint is negative after it dissolves • Model does not consider the temperature in the mouth • Model does not consider rate of saliva production • Mint could be swallowed before it dissolves in the mouth	B1	3.5b
		(1)	
			(8 marks)

Notes for Question 10	
(a)	
M1:	Translates the description of the model into mathematics. See scheme.
M1:	Separates the variables of their differential equation which is in the form $\frac{\mathrm{d}r}{\mathrm{d}t} = \mathrm{f}(r)$ and some attempt at integration. (e.g. attempts to integrate at least one side). e.g. $\int r^2 \,\mathrm{d}r = \int \pm k \,\mathrm{d}t$ and some attempt at integration. Condone the lack of integral signs
Note:	You can imply the M1 mark for $r^2 dr = -k\,\mathrm{d}t \Rightarrow \frac{1}{3}r^3 = -kt$
Note:	A numerical value of k (e.g. $k = \pm 1$) is allowed for the first two M marks
A1:	Correct integration to give $\frac{1}{3}r^3 = \pm kt$ with or without a constant of integration, c
M1:	For a complete process of using the boundary conditions to find both their unknown constants and finds an equation linking r and t So applies either • $t = 0, r = 5$ and $t = 4, r = 3,$ or • $t = 0, r = 5$ and $t = 240, r = 3,$ ***on their integrated equation*** to find their constants k and c ***and*** obtains an equation linking r and t
A1:	Correct equation, with variables r and t fully defined including correct reference to units. • $\frac{1}{3}r^3 = -\frac{49}{6}t + \frac{125}{3}$, {or an equivalent equation,} where r, in mm, is the radius {of the mint} and t, in minutes, is the time from when it {the mint} was placed in the mouth • $\frac{1}{3}r^3 = -\frac{49}{360}t + \frac{125}{3}$, {or an equivalent equation,} where r, in mm, is the radius {of the mint} and t, in seconds, is the time from when it {the mint} was placed in the mouth
Note:	Allow correct equations such as • in minutes, $r = \sqrt[3]{\frac{250 - 49t}{2}}$, $r^3 = -\frac{49}{2}t + 125$ or $t = \frac{250 - 2r^3}{49}$ • in seconds, $r = \sqrt[3]{\frac{15000 - 49t}{120}}$, $r^3 = -\frac{49}{120}t + 125$ or $t = \frac{15000 - 120r^3}{49}$
Note:	t defined as "the time from the start" is not sufficient for the final A1
(b)	
M1:	Sets $r = 0$ in their part (a) equation which links r with t and rearranges to make $t = ...$
A1:	5 minutes 6 seconds cao (**Note:** 306 seconds with no reference to 5 minutes 6 seconds is A0)
Note:	Give M0 if their equation would solve to give a negative time or a negative time is found
Note:	You can mark part (a) and part (b) together
(c)	
B1:	See scheme
Note:	Do not accept by itself • mint may not dissolve at a constant rate • rate of decrease of mint must be constant • $0 \le t < \frac{250}{49}$, $r \ge 0$; without any written explanation • reference to a mint having $r > 5$

Question	Scheme	Marks	AOs
	$1+11x-6x^2 \equiv \quad B \quad C$		