adamjee maths

A-LEVEL MATHS / EDEXCEL / FREE RESOURCES

Small angle approximations

Mark scheme

5 questions

Exam questions and marking instructions remain credited to their board.

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Question	Scheme	Marks	AOs
6(a)	Attempts $\cos^2(2x) \approx \left(1-\frac{(2x)^2}{2}\right)^2 = ...\left(1-4x^2+4x^4\right)$	M1	1.1b
	$1-\cos^2(2x) \approx 1-\left(1-4x^2+4x^4\right) = 4x^2-4x^4$ *	A1*	1.1b
		(2)	
(b)	$\frac{1-\cos^2(2x)}{\sin\left(\frac{x}{3}\right)\tan\left(\frac{x}{2}\right)} \approx \frac{4x^2-4x^4}{\left(\frac{x}{3}\right)\left(\frac{x}{2}\right)}$	M1	1.1b
	$= \frac{6}{x^2}\left(4x^2-4x^4\right) = 24-24x^2$	A1	2.1
		(2)	
(c)	24	B1ft	2.2a
	If x is (very) small, then any terms in x^2 are negligible.	dB1ft	2.4
		(2)	
			(6 marks)

Notes

(a)

M1: Attempts to use $\cos\theta \approx 1-\frac{\theta^2}{2}$ with θ, 2θ, x or $2x$ and squares the resulting expression.

Condone poor squaring, e.g., $\left(1-2x^2\right)^2 = 1-4x^4$ or missing brackets $\left(1-\frac{2x^2}{2}\right)^2 = \left(1-x^2\right)^2 = ...$

but not e.g. $\cos 2x \approx 2\left(1-\frac{x^2}{2}\right)$

May be implied by e.g. $1-\left(1-2x^2\right)^2 = 2x^2\left(2-2x^2\right)$ from difference of two squares.

A1*: Correct proof with an intermediate line such as $1-\left(1-4x^2+4x^4\right)$

Do not be concerned if the LHS does not appear and ignore any spurious = 0 in their work.

There should be no obvious incorrect statements in the proof e.g. $1-\left(1+4x^2-4x^4\right)$

Do not condone incorrect work including invisible brackets such as $1-1-4x^2+4x^4$ or $\left(1-\frac{2x^2}{2}\right)$ but condone a missing trailing bracket e.g. $1-(1-4x^2+4x^4$

Condone an attempt that starts in stages, which may not deal with the full expression e.g., $1-\left(1-\frac{4x^2}{2}\right) \rightarrow 1-\left(1-2x^2\right)^2 = ...$ has a missing square on the first bracket but is recovered in the next stage/step.

The final line should be in terms of x but condone a slip to e.g. θ in the workings of the proof.

If they complete the proof using θ they must revert to x to score this mark.

Special Cases: In each of these cases below, please send to review.

You may see attempts that use e.g.

- $\cos 2x = \pm 2\cos^2 x \pm 1$ or $\cos^2 2x = \pm\frac{1}{2} \pm \frac{1}{2}\cos 4x$
- $\cos 2x = \pm 1 \pm 2\sin^2 x$ followed by the small angle approximation for sin x
- Maclaurin expansions

(b)

M1: Uses the **given answer** to part (a) and writes $\sin\left(\frac{x}{3}\right)$ **and** $\tan\left(\frac{x}{2}\right)$ using correct small angle approximations. Only allow misreads that are clearly misreads e.g. they cannot replace $\tan\left(\frac{x}{2}\right)$ with $\frac{x}{3}$ unless $\tan\left(\frac{x}{3}\right)$ is seen first. In such cases they will lose the A mark in (b) but both of the marks in (c) are available. Condone mixed variables for this mark.

A1: $24-24x^2$ but condone e.g. $24+-24x^2$ or e.g. $a = 24$ and $b = -24$
Condone recovery of missing/invisible brackets but the work must otherwise be correct.
Do not condone mixed variables being recovered unless explicitly replaced with x before cancelling x^2.

No marks are scored in (b) for using the Maclaurin expansions for $\sin\left(\frac{x}{3}\right)$ and/or $\tan\left(\frac{x}{2}\right)$

(c)

B1ft: 24 but this must follow from the non-zero constant term in their answer to (b).
Do not allow e.g. $x = 24$
Allow follow through on their non-zero constant term from a polynomial in x.

dB1ft: Suitable reason given but it should refer to their x^2 term in some way (and any additional terms if they have any). Ignore spurious remarks e.g. "if $x < 1$" unless contradictory.
Dependent on the previous B1ft mark.
Some acceptable examples:

- "24"$x^2 \to 0$ or $x^2 \to 0$
- "We can ignore the x^2 term"
- "x^2 is much smaller than 24"
- As $x \to 0$ their $a + bx^2 \to a$ (condone as $x \to 0$, bx^2 "becomes" 0)
- $\lim_{x\to 0} a+bx^2 = \left(a+b(0)^2\right) = a$
- Since x is very small, $24 - 24(0)^2 = 24$ (and condone if the squared is missing).

They cannot *just* substitute in 0 or a very small value for x to score the mark for the reason.
A reason such as "the answer rounds to 24" is not acceptable.
There must be some justification, either "as $x \to 0$" or "since x is very small"
so e.g. "$24 - 24(0)^2 = 24$ on its own scores B1ft dB0ft
This mark may be scored from any polynomial in x that includes at least a constant term and an x^2 term.

They cannot go back to e.g. $4x^4$ or $\frac{4x^4}{\frac{1}{6}x^2}$ and say this is negligible or $\to 0$ as they haven't dealt with the x in the denominator.

Question	Scheme	Marks	AOs

5	One of $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\frac{(3\theta)^2}{2}\right)$ or equivalents.	B1	1.1a
	$\dfrac{\theta\tan 2\theta}{1-\cos 3\theta} = \dfrac{\theta\times 2\theta}{1-\left(1-\frac{(3\theta)^2}{2}\right)}$	M1	2.1
	$=\dfrac{4}{9}$ or exact equivalent.	A1	1.1b
		(3)	

(3 marks)

Notes

B1: Award this mark for $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\dfrac{(3\theta)^2}{2}\right)$ or equivalents.

May be seen when working on numerator or denominator separately or within the fraction.

This is a B mark so if awarding for $\cos 3\theta$ do not condone missing brackets e.g. $1-\dfrac{3\theta^2}{2}$ unless they are recovered or are implied by subsequent work.

M1: Attempts to use both correct small angle approximations in the given expression.

For this mark they must have attempted to use $\tan 2\theta = 2\theta$ and $\cos 3\theta = 1-\dfrac{(3\theta)^2}{2}$ in the given expression but condone poor bracketing e.g. $\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}$ or e.g. $\dfrac{\theta\times 2\theta}{1-1-\frac{3\theta^2}{2}}$

Do not allow e.g. $\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$ as this suggests they are approximating $\dfrac{\theta\tan 2\theta}{\cos 3\theta}$

A1: Correct value. Do not allow rounded decimals e.g. 0.444 but allow if recurring decimals are clearly indicated e.g. $0.\dot{4}$ Do not allow e.g. $\dfrac{2}{4.5}$. Ignore any units if given.

Isw once a correct answer is seen.

Examples:

$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \dfrac{4}{3}\left(\text{or } -\dfrac{4}{3}\right)$	scores B1M1A0 (Missing brackets not recovered)
$\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$	scores B1M0A0 (Missing "1 –" in the denominator so M0)
$\dfrac{\theta\times 2\theta}{1+\left(1-\frac{(3\theta)^2}{2}\right)}$	scores B1M0A0 (Has "1 +" in the denominator so M0)
$\dfrac{\theta\times\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \ldots$	scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)
$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)} = \dfrac{2\theta^2}{\frac{9\theta^2}{2}} = \dfrac{2}{9}$	scores B1M1A0 (Missing brackets recovered)
$\dfrac{\theta\times 2\theta}{1-\left(1-\left(\frac{3\theta}{2}\right)^2\right)}$	Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)

$\frac{\theta\times 2\theta}{\frac{9\theta^2}{2}}=\frac{2}{18}$	B1M1A0 (The B1 is awarded for the numerator but can be implied by the denominator. The M1 is implied)
$\frac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}=\frac{4}{9}$	B1M1A1 (The correct value implies correct recovery of missing brackets.)

Note that other approaches are possible using identities.
In such cases we will allow <u>correct</u> work leading to an expression that if terms in θ^3 and higher can be ignored will lead to $\frac{4}{9}$
But to score the M mark they must be using correct identities and correct approximations but condone bracketing errors as in the main scheme.

Examples:

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\frac{\sin 2\theta}{\cos 2\theta}}{1-\cos 3\theta}=\frac{\theta\times\frac{2\theta}{1-\frac{(2\theta)^2}{2}}}{1-\left(1-\frac{(3\theta)^2}{2}\right)}=\frac{2\theta^2}{1-2\theta^2}\times\frac{2}{9\theta^2}=\frac{4\theta^2}{9\theta^2-18\theta^4}$$

$$=\frac{4\theta^2}{9\theta^2}=\frac{4}{9}$$

Scores B1M1A1

Similarly: $\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\sin 2\theta}{\cos 2\theta(1-\cos 3\theta)}=\frac{\theta\times 2\theta}{\left(1-\frac{(2\theta)^2}{2}\right)\left(1-\left(1-\frac{(3\theta)^2}{2}\right)\right)}=\frac{2\theta^2}{1-2\theta^2}\times\frac{2}{9\theta^2}$ etc.

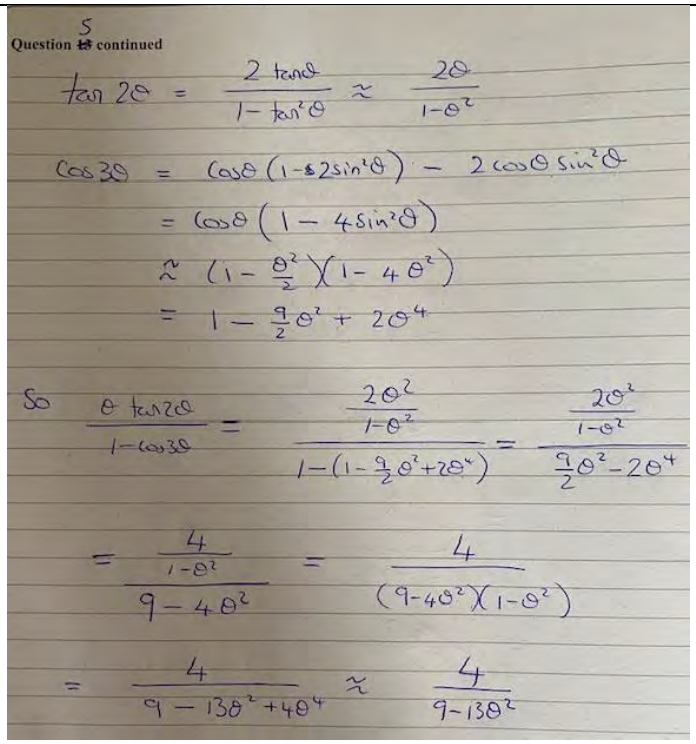
$$\frac{4\theta^2}{9\theta^2-18\theta^4}=\frac{4\theta^2}{9\theta^2}=\frac{4}{9}$$

Scores B1M1A1

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\sin 2\theta}{\cos 2\theta(1-\cos 3\theta)}=\frac{\theta\times 2\theta}{\left(1-\frac{(2\theta)^2}{2}\right)\left(1-\left(1-\frac{(3\theta)^2}{2}\right)\right)}=\frac{2\theta^2}{1-2\theta^2}\times\frac{2}{9\theta^2}$$

$$=\frac{4\theta^2}{9\theta^2-18\theta^4}=\frac{4}{9-18\theta^2}=\frac{4}{9}$$

Scores B1M1A0
(They cannot just assume the term in θ^2 is 0 unless they provide a convincing limiting argument e.g. $\lim_{\theta\to 0}\frac{4}{9-18\theta^2}=\frac{4}{9}$ or equivalent)



Scores B1M1A0

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta} = \frac{\theta\times\dfrac{2\tan\theta}{1-\tan^2\theta}}{1-\left(4\cos^3\theta-3\cos\theta\right)} = \frac{\theta\times\dfrac{2\theta}{1-\theta^2}}{1-\left(4\left(1-\dfrac{\theta^2}{2}\right)^3-3\left(1-\dfrac{\theta^2}{2}\right)\right)}$$

Scores B1M1A0

Note that attempts to use expansions in higher powers of θ should be sent to review.

Question	Scheme	Marks	AOs

4	**Examples:** $4\sin\frac{\theta}{2} \approx 4\left(\frac{\theta}{2}\right),\ 3\cos^2\theta \approx 3\left(1-\frac{\theta^2}{2}\right)^2$ $3\cos^2\theta = 3\left(1-\sin^2\theta\right) \approx 3\left(1-\theta^2\right)$ $3\cos^2\theta = 3\frac{\left(\cos 2\theta+1\right)}{2} \approx \frac{3}{2}\left(1-\frac{4\theta^2}{2}+1\right)$	M1	1.1a
	Examples: $4\sin\frac{\theta}{2}+3\cos^2\theta \approx 4\left(\frac{\theta}{2}\right)+3\left(1-\frac{\theta^2}{2}\right)^2$ $4\sin\frac{\theta}{2}+3\cos^2\theta = 4\left(\frac{\theta}{2}\right)+3\left(1-\sin^2\theta\right) \approx 2\theta+3\left(1-\theta^2\right)$ $4\sin\frac{\theta}{2}+3\cos^2\theta = 4\sin\frac{\theta}{2}+3\frac{\left(\cos 2\theta+1\right)}{2} \approx 4\left(\frac{\theta}{2}\right)+\frac{3}{2}\left(1-\frac{4\theta^2}{2}+1\right)$	dM1	1.1b
	$= 2\theta+3\left(1-\theta^2+\ldots\right) = 3+2\theta-3\theta^2$	A1	2.1
		(3)	
			(3 marks)

Notes

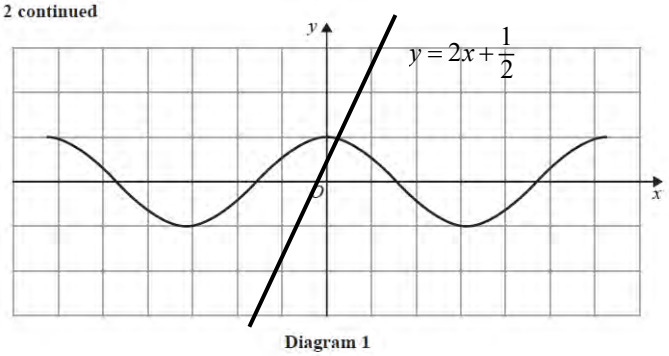
M1: Attempts to use at least one correct approximation **within the given expression**.

Either $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$ or $\cos\theta \approx 1-\frac{\theta^2}{2}$ or e.g. $\sin\theta \approx \theta$ if they write $\cos^2\theta$ as $1-\sin^2\theta$ or e.g. $\cos 2\theta \approx 1-\frac{\left(2\theta\right)^2}{2}$ (condone missing brackets) if they write $\cos^2\theta$ as $\frac{1+\cos 2\theta}{2}$.

Allow sign slips only with any identities used but the appropriate approximations must be applied.

dM1: Attempts to use correct approximations **with the given expression** to obtain an expression in terms of θ only. **Depends on the first method mark.**

A1: Correct terms following correct work. Allow the terms in any order and ignore any extra terms if given correct or incorrect.

Question	Scheme	Marks	AOs
2(a)	2 continued  Diagram 1	B1	3.1a
	For an allowable linear graph and explaining that there is only one intersection	B1	2.4
		(2)	
(b)	$\cos x-2x-\frac{1}{2}=0\Rightarrow 1-\frac{x^2}{2}-2x-\frac{1}{2}=0$	M1	1.1b
	Solves their $x^2+4x-1=0$	dM1	1.1b
	Allow awrt 0.236 but accept $-2+\sqrt{5}$	A1	1.1b
		(3)	
			(5 marks)

(a)

B1: Draws $y=2x+\frac{1}{2}$ on Figure 1 or Diagram 1 with an attempt at the correct gradient and the correct intercept. Look for a straight line with an intercept at $\approx\frac{1}{2}$ and a further point at $\approx\left(\frac{1}{2},1\frac{1}{2}\right)$ Allow a tolerance of 0.25 of a square in either direction on these two points. It must appear in quadrants 1, 2 and 3.

B1: There must be an allowable linear graph on Figure 1 or Diagram1 for this to be awarded
Explains that as there is only one intersection so there is just one root.
This requires a reason and a minimal conclusion.
The question asks candidates to explain but as a bare minimum allow one "intersection"

Note: An allowable linear graph is one with intercept of $\pm\frac{1}{2}$ with one intersection with $\cos x$ **OR** gradient of ± 2 with one intersection with $\cos x$

(b)

M1: Attempts to use the small angle approximation $\cos x=1-\frac{x^2}{2}$ in the given equation.

The equation must be in a single variable but may be recovered later in the question.

dM1: Proceeds to a 3TQ in a single variable and attempts to solve. See General Principles
The previous M must have been scored. Allow completion of square or formula or calculator. Do not allow attempts via factorisation unless their equation does factorise. You may have to use your calculator to check if a calculator is used.

A1: Allow $-2+\sqrt{5}$ or awrt 0.236.

Do not allow this where there is another root given and it is not obvious that 0.236 has been chosen.

Question	Scheme	Marks	AOs
1	Attempts either $\sin 3\theta \approx 3\theta$ or $\cos 4\theta \approx 1-\frac{(4\theta)^2}{2}$ in $\frac{1-\cos 4\theta}{2\theta \sin 3\theta}$	M1	1.1b
	Attempts both $\sin 3\theta \approx 3\theta$ and $\cos 4\theta \approx 1-\frac{(4\theta)^2}{2} \rightarrow \frac{1-\left(1-\frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$ and attempts to simplify	M1	2.1
	$=\frac{4}{3}$ oe	A1	1.1b
		(3)	

(3 marks)

M1: Attempts either $\sin 3\theta \approx 3\theta$ or $\cos 4\theta \approx 1-\frac{(4\theta)^2}{2}$ in the given expression.

See below for description of marking of $\cos 4\theta$

M1: Attempts to substitute both $\sin 3\theta \approx 3\theta$ and $\cos 4\theta \approx 1-\frac{(4\theta)^2}{2}$

$\rightarrow \frac{1-\left(1-\frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$ and attempts to simplify.

Condone missing bracket on the 4θ so $\cos 4\theta \approx 1-\frac{4\theta^2}{2}$ would score the method

Expect to see it simplified to a single term which could be in terms of θ

Look for an answer of k but condone $k\theta$ following a slip

A1: Uses both identities and simplifies to $\frac{4}{3}$ or exact equivalent with no incorrect lines BUT allow recovery on missing bracket for $\cos 4\theta \approx 1-\frac{4\theta^2}{2}$.

Eg. $\frac{1-\left(1-\frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}=\frac{8\theta^2}{6\theta}=\frac{4}{3}$ is M1 M1 A0

Condone awrt 1.33.

.....

Alt: $\frac{1-\cos 4\theta}{2\theta \sin 3\theta}=\frac{1-\left(1-2\sin^2 2\theta\right)}{2\theta \sin 3\theta}=\frac{2\sin^2 2\theta}{2\theta \sin 3\theta}=\frac{2\times(2\theta)^2}{2\theta \times 3\theta}=\frac{4}{3}$

M1 For an attempt at $\sin 3\theta \approx 3\theta$ or the identity $\cos 4\theta = 1-2\sin^2 2\theta$ with $\sin 2\theta \approx 2\theta$

M1 For both of the above and attempts to simplify to a single term.

A1 $\frac{4}{3}$ oe